Language Modeling and the Noisy Channel

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Outline

The Noisy Channel

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LM Smoothing and the EM Algorithm

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Prototypical case:

	Input	**Channel**	**Output (noisy)**
	0,1,1,1,0,1,0,1,...	$\xrightarrow{\text{adds noise}}$	0,1,1,0,0,1,1,0,...

Model: probability of error (noise):

$$p(0|1) = 0.3 \quad p(1|1) = 0.7 \quad p(1|0) = 0.4 \quad p(0|0) = 0.6$$

The Task:

- *known:* the noisy output
- *want to know:* the input (decoding)

Noisy Channel Applications

- **OCR** — straightforward: text $\rightarrow$ print (adds noise), scan image $\rightarrow$ image
- **Handwriting recognition** — text $\rightarrow$ neurons, muscles ("noise"), scan/digitize $\rightarrow$ image
- **Speech recognition** (dictation, commands, ...) — text $\rightarrow$ conversion to acoustic signal ("noise") $\rightarrow$ acoustic waves
- **Machine Translation** — text in target language $\rightarrow$ translation ("noise") $\rightarrow$ source language

Nowadays, SoTA systems are end-to-end neural/LLM-based.

Bayes Rule for Noisy Channel

Recall Bayes Rule:

$$p(A|B) = \frac{p(B|A)\,p(A)}{p(B)}$$

The Golden Rule:

$$A_{\text{best}} = \arg\max_{A}\ p(B|A)\,p(A)$$

- $p(B|A)$: **acoustic/image/translation** application-specific name
- $p(A)$: the **language model**

Language Models

Outline

Probability of a Sentence

- Sequence of word forms: $W = (w_1, w_2, w_3, \ldots, w_d)$
- **The big modeling question:** $p(W) =?$
- By the chain rule (Bayes):

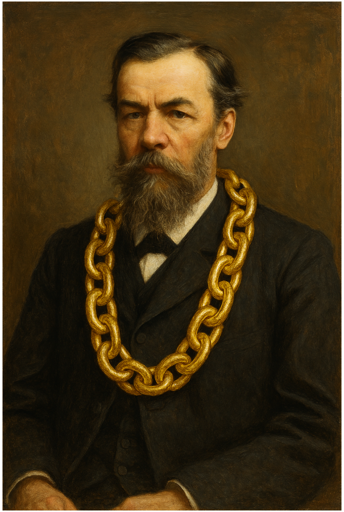
$$p(W) = p(w_1) \cdot p(w_2|w_1) \cdot p(w_3|w_1, w_2) \cdots p(w_d|w_1, \ldots, w_{d-1})$$

- Direct estimation is **not practical:** even short W requires too many parameters

Markov Chain

- **Unlimited memory** (cf. previous slide):
  - for w_i, we know all predecessors $w_1, w_2, \ldots, w_{i-1}$
- **Limited memory** (kth order Markov approximation):
  - disregard "too old" predecessors
  - remember only k previous words: $w_{i-k}, \ldots, w_{i-1}$
- **Stationary character** (no change over time):

$$p(W) \approx \prod_{i=1}^{d} p(w_i \mid w_{i-k}, w_{i-k+1}, \ldots, w_{i-1})$$



n-gram Language Models

$(n-1)$th order Markov approximation $\Rightarrow$ n-gram LM:

$$p(W) \stackrel{\text{def}}{=} \prod_{i=1}^{d} p(w_i \mid w_{i-n+1}, \ldots, w_{i-1})$$

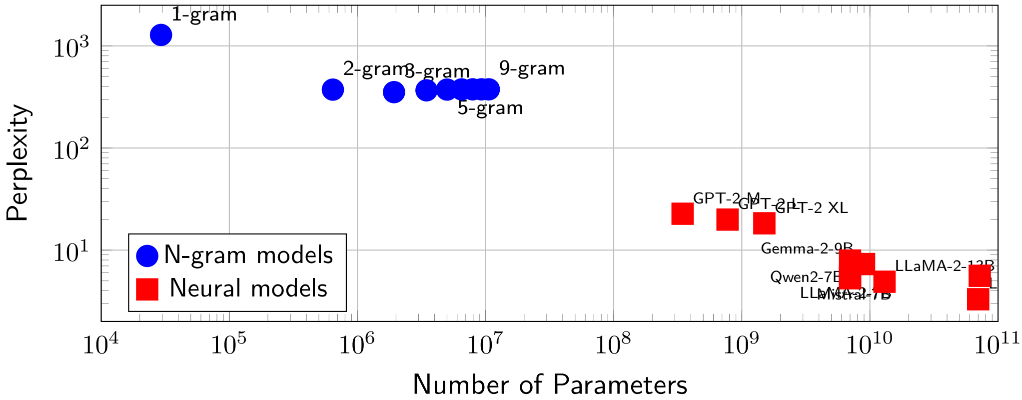
With vocabulary $|V| = 60\,000$:

Model	**Name**	**Parameters**
0-gram	uniform	1
1-gram	unigram	6×10^4
2-gram	bigram	3.6×10^9
3-gram	trigram	2.16×10^{14}

Will it be that many parameters in practice?

No, the count tables will be very sparse.

Number of parameters vs. perplexity on the Wiki



Not a fair comparison: with more data, the n-gram models would also slightly improve.

Maximum Likelihood Estimate

MLE = Relative Frequency — best predicts the training data T

For trigrams from training data T:

- Count $c_3(w_{i-2}, w_{i-1}, w_i)$ — sequences of three words in T
- Count $c_2(w_{i-1}, w_i)$ — sequences of two words in T
  - either $c_2(y, z) = \sum_w c_3(y, z, w)$
  - or count separately at beginning/end of data

MLE for trigrams:

$$\hat{p}(w_i \mid w_{i-2}, w_{i-1}) = \frac{c_3(w_{i-2}, w_{i-1}, w_i)}{c_2(w_{i-2}, w_{i-1})}$$

Training data:

`<s> <s> He can buy the can of soda.`

- **Unigram:** $p_1(\text{He}) = p_1(\text{buy}) = \cdots = 0.125; \quad p_1(\text{can}) = 0.25$
- **Bigram:** $p_2(\text{He}|\texttt{<s>}) = 1,\ p_2(\text{can}|\text{He}) = 1,\ p_2(\text{buy}|\text{can}) = 0.5, \ldots$
- **Trigram:** $p_3(\text{He}|\texttt{<s>},\texttt{<s>}) = 1,\ p_3(\text{can}|\texttt{<s>}, \text{He}) = 1, \ldots$
- **Entropy:** $H(p_1) = 2.75, \quad H(p_2) = 0.25, \quad H(p_3) = 0 \quad \rightarrow$ Great?!

No, the model is **overfitted** to the training data.

LM: An Example — The Problem

Cross-entropy on:

$S =$ `<s> <s> It was the greatest buy of all.`

Even $H_S(p_1) = \infty$ (and $= H_S(p_2) = H_S(p_3) = \infty$), because:

- all unigrams except $p_1(\text{the}), p_1(\text{buy}), p_1(\text{of}), p_1(.)$ are 0
- all bigram probabilities are 0
- all trigram probabilities are 0

Goal: make all (theoretically possible) probabilities non-zero.

Try our character-level LM as autocomplete!

https://ufallab.ms.mff.cuni.cz/~libovicky/ngram_predictor.html

LM Smoothing and the EM Algorithm

Outline

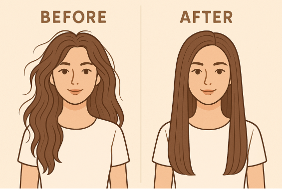
The Zero Problem

- "Raw" n-gram LM has necessarily some zeros
  - trigram model: $\sim 2.16 \times 10^{14}$ parameters; data $\sim 10^{9}$ words
- **Which are true zeros?**
  - Optimal: every trigram seen several times — impossible in practice
  - We don't know; we *must* eliminate the zeros
- **Two kinds of zeros:**
  - $p(w|h) = 0$ (history seen, word not)
  - $p(h) = 0$ (history itself unseen)

Why Do We Need Non-Zero Probabilities?

- **Avoid infinite cross-entropy:**
  - occurs when test data contains an event not seen in training
  - $H(p) = \infty$ prevents any meaningful comparison
- **Make the system more robust:**
  - Low-count estimates: detailed but less reliable
  - High-count estimates: reliable but less detailed

Smoothing — General Idea



Obtain new $p'(w)$ (same support): almost $p(w)$ but no zeros.

1. **Discount** some $p(w) > 0$: new $p'(w) < p(w)$

$$\sum_{w \text{ discounted}} \big(p(w) - p'(w)\big) = D$$

2. **Redistribute** D to all w with $p(w) = 0$ (possibly also to low-probability words)
3. Ensure $\sum_w p'(w) = 1$

Smoothing by Adding 1 (Laplace)

$$p'(w|h) = \frac{c(h,w) + 1}{c(h) + |V|}$$

Problem: if $|V| \gg c(h)$, too much probability mass is redistributed.

Example: Training: `<s> what is it what is small ?` $(|T| = 8, |V| = 12)$

- $p(\text{what}) = 0.25,\ p(\text{it}) = 0.125,\ p(.) = 0$
- $p'(\text{what}) \approx 0.15,\ p'(\text{it}) \approx 0.10,$
  $p'(.) \approx 0.05$

Adding Less Than 1 (δ-smoothing)

$$p'(w|h) = \frac{c(h,w) + \delta}{c(h) + \delta|V|}, \quad \delta \in (0,1)$$

Example with $\delta = 0.1$:

- p'(what) ≈ 0.23, p'(it) ≈ 0.12, p'(.) ≈ 0.01
- p'(it is flying.) ≈ 0.000003 (was 0 with raw model)

Linear Interpolation Smoothing

Combine distributions of various detail vs. reliability:

$$p'(w_i \mid w_{i-2}, w_{i-1}) = \lambda_3\, p_3 + \lambda_2\, p_2 + \lambda_1\, p_1 + \lambda_0\, \frac{1}{|V|}$$

with $\lambda_j > 0$ and $\sum_j \lambda_j = 1$.

- High λ_3: detailed but less reliable (sparse)
- High λ_1: reliable but less detailed

Estimating the λ Weights

If we use the training data, the best thing to do is to set $\lambda_3 = 1$ (MLE).
What shall we do?

- **Minimize cross-entropy on held-out data** H (not training data!):

$$-\frac{1}{|H|}\sum_{i=1}^{|H|}\log_2 p'(w_i|h_i)$$

- Why not training data? Using T always gives $\lambda_3 = 1$ (trivially minimized by MLE).
- Split data into: **training** T, **held-out** H, **test** S

The EM Algorithm for Smoothing

E-step — Expected counts:

$$c(\lambda_j) = \sum_{i=1}^{|H|} \frac{\lambda_j \, p_j(w_i|h_i)}{p'(w_i|h_i)}$$

M-step — Update:

$$\lambda_j^{\text{new}} = \frac{c(\lambda_j)}{\sum_k c(\lambda_k)}$$

Algorithm:

1. Start with all $\lambda_j > 0$
2. Compute expected counts (E-step)
3. Update λs (M-step)
4. Repeat until convergence: $|\lambda_j - \lambda_j^{\text{new}}| < \varepsilon$

Convergence guaranteed by Jensen's inequality.
$\log(\mathcal{E}[X]) \leq \mathcal{E}[\log(X)]$
$\rightarrow$ EM never decreases likelihood.

Simple EM Example

Unigram: $p(a) = 0.25,\ \ p(b) = 0.5,\ \ p(c \ldots r) = 1/64;$ **Heldout:** `baby`

Start: $\lambda_1 = 0.5$

$$p'(b) = 0.5 \times 0.5 + 0.5/26 \approx 0.27$$

Expected counts:

$$c(\lambda_1) \approx 2.72, \quad c(\lambda_0) \approx 1.28$$

Update: $\lambda_1^{\text{new}} \approx 0.68,\ \ \lambda_0^{\text{new}} \approx 0.32$

Repeat until $|\lambda_j - \lambda_j^{\text{new}}| < 0.01.$

Summary

- The **noisy channel model** underlies OCR, ASR, MT, and POS tagging; it requires a language model $p(A)$.
- n-gram LMs approximate $p(W)$ via a Markov assumption; estimated by MLE (relative frequency).
- Raw n-gram models suffer from **zero probabilities**; smoothing is essential.
- **Linear interpolation** with back-off to lower-order models is a principled solution.
- λ weights are estimated on **held-out data** using the **EM algorithm**.

https://ufal.mff.cuni.cz/courses/npfl124