Introduction to Deep Learning

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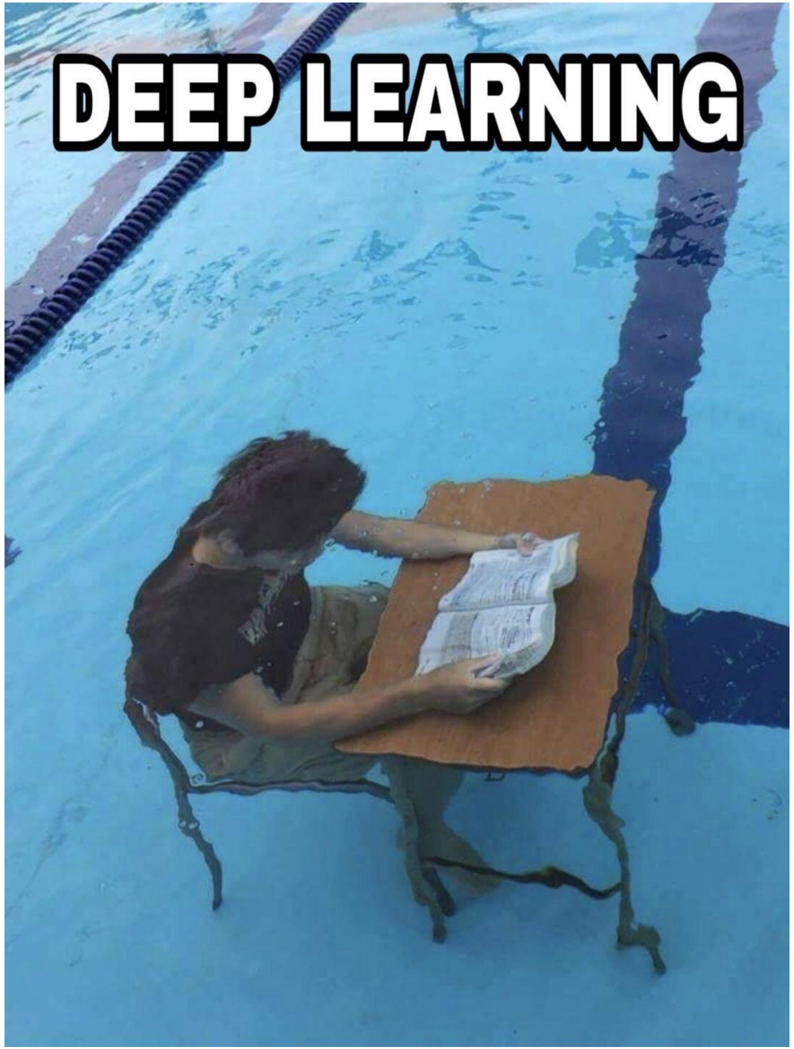
LANGTECH

EUROPEAN UNION
European Structural and Investment Fund
Operational Programme Research,
Development and Education

Charles University in Prague
Faculty of Mathematics and Physics
Institute of Formal and Applied Linguistics

What is Deep Learning



https://i.redd.it/t87gswsbmnq41.jpg

Deep Learning Highlights

- Image recognition
- Object detection
- Image segmentation
- Human pose estimation
- Image labeling
- Visual question answering
- Speech recognition and generation
- Lip reading
- Machine translation
- Machine translation without parallel data
- Chess, Go and Shogi
- Multiplayer Capture the flag
- StarCraft II

Organization

Course Website: https://ufal.mff.cuni.cz/courses/npfl114

- Slides, recordings, assignments, Exam questions

Course Repository: https://github.com/ufal/npfl114

- Templates for the assignments, slide sources.

Piazza

- Piazza will be used as a communication platform.

  You can post questions or notes,
  - privately to the instructors, or
  - to everyone (signed or anonymously).

  Students can answer other student's questions too, which allows you to get faster response. Please do not send complete solutions to other students, only excerpts of the source files.
- Please use Piazza for **all communication** with the instructors.
- You will get the invite link after the first lecture.

ReCodEx

https://recodex.mff.cuni.cz

- The assignments will be evaluated automatically in ReCodEx.
- If you have a MFF SIS account, you will be able to create an account using your CAS credentials and should automatically see the right group.
- Otherwise, there will be **instructions** on **Piazza** how to get ReCodEx account (generally you will need to send me a message with several pieces of information and I will send it to ReCodEx administrators in batches).

Course Requirements

Practicals

- There will be 2-3 assignments a week, each with a 2-week deadline.
  - There is also another week-long second deadline, but for less points.

- After solving the assignment, you get non-bonus points, and sometimes also bonus points.
- To pass the practicals, you need to get 80 non-bonus points. There will be assignments for at least 120 non-bonus points.
- If you get more than 80 points (be it bonus or non-bonus), they will be all transferred to the exam. Additionally, if you solve all the assignments, you pass the exam with grade 1.

Lecture

You need to pass a written exam (or solve all the assignments).

- All questions are publicly listed on the course website.
- There are questions for 100 points in every exam, plus the surplus points from the practicals and plus at most 10 surplus points for **community work** (improving slides, …).
- You need 60/75/90 points to pass with grade 3/2/1; 75 points for PhD students.

Notation

- a, $\boldsymbol{a}$, $\boldsymbol{A}$, A: scalar (integer or real), vector, matrix, tensor
  - all vectors are always **column** vectors
  - transposition changes a column vector into a row vector, so $\boldsymbol{a}^T$ is a row vector
  - we denote the **dot (scalar) product** of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ using $\boldsymbol{a}^T\boldsymbol{b}$
    - we understand it as matrix multiplication
  - the $\|\boldsymbol{a}\|_2$ or just $\|\boldsymbol{a}\|$ is the Euclidean (or L^2) norm
    - $\|\boldsymbol{a}\|_2 = \sqrt{\sum_i a_i^2}$
- a, $\mathbf{a}$, $\mathbf{A}$: scalar, vector, matrix random variable
- $\frac{df}{dx}$: derivative of f with respect to x
- $\frac{\partial f}{\partial x}$: partial derivative of f with respect to x
- $\nabla_{\boldsymbol{x}} f(\boldsymbol{x})$: gradient of f with respect to $\boldsymbol{x}$, i.e., $\left(\frac{\partial f(\boldsymbol{x})}{\partial x_1}, \frac{\partial f(\boldsymbol{x})}{\partial x_2}, \ldots, \frac{\partial f(\boldsymbol{x})}{\partial x_n}\right)$

Random Variables

A random variable x is a result of a random process, and it can be either discrete or continuous.

Probability Distribution

A probability distribution describes how likely are the individual values that a random variable can take.

The notation $\mathrm{x} \sim P$ stands for a random variable x having a distribution P.

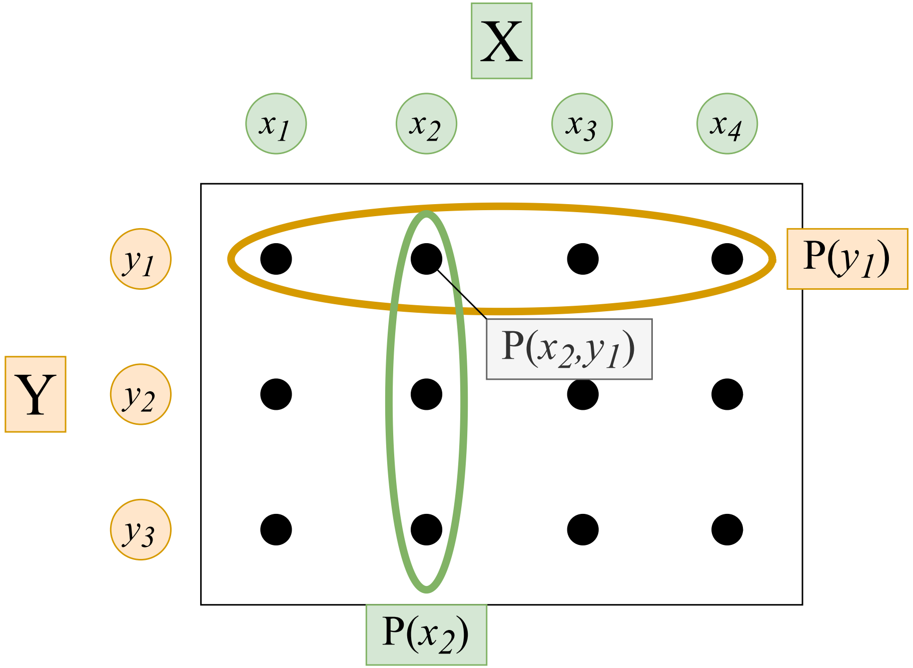
For discrete variables, the probability that x takes a value x is denoted as $P(x)$ or explicitly as $P(\mathrm{x} = x)$. All probabilities are nonnegative, and the sum of the probabilities of all possible values of x is $\sum_x P(\mathrm{x} = x) = 1$.

For continuous variables, the probability that the value of x lies in the interval $[a, b]$ is given by $\int_a^b p(x)\,\mathrm{d}x$, where $p(x)$ is the *probability density function*, which is always nonnegative and integrates to 1 over the range of all values of x.

Joint, Conditional, Marginal Probability

For two random variables, a **joint probability distribution** is a distribution of all possible pairs of outputs (and analogously for more than two):

$$P(\mathrm{x} = x_2, \mathrm{y} = y_1).$$



Marginal distribution is a distribution of one (or a subset) of the random variables and can be obtained by summing over the other variable(s):

$$P(\mathrm{x} = x_2) = \sum\nolimits_y P(\mathrm{x} = x_2, \mathrm{y} = y).$$

Conditional distribution is a distribution of one (or a subset) of the random variables, given that another event has already occurred:

$$P(\mathrm{x} = x_2 | \mathrm{y} = y_1) = P(\mathrm{x} = x_2, \mathrm{y} = y_1) / P(\mathrm{y} = y_1).$$

If $P(\mathrm{x}, \mathrm{y}) = P(\mathrm{x}) \cdot P(\mathrm{y})$ or $P(\mathrm{x}|\mathrm{y}) = P(\mathrm{x})$, random variables x and y are **independent**.

Random Variables

Expectation

The expectation of a function $f(x)$ with respect to a discrete probability distribution $P(\mathrm{x})$ is defined as:

$$\mathbb{E}_{\mathrm{x}\sim P}[f(x)] \stackrel{\text{def}}{=} \sum_x P(x)f(x).$$

For continuous variables, the expectation is computed as:

$$\mathbb{E}_{\mathrm{x}\sim p}[f(x)] \stackrel{\text{def}}{=} \int_x p(x)f(x)\,\mathrm{d}x.$$

If the random variable is obvious from context, we can write only $\mathbb{E}_P[x]$, $\mathbb{E}_{\mathrm{x}}[x]$, or even $\mathbb{E}[x]$.

Expectation is linear, i.e., for constants $\alpha, \beta \in \mathbb{R}$:

$$\mathbb{E}_{\mathrm{x}}[\alpha f(x) + \beta g(x)] = \alpha\mathbb{E}_{\mathrm{x}}[f(x)] + \beta\mathbb{E}_{\mathrm{x}}[g(x)].$$

Variance

Variance measures how much the values of a random variable differ from its mean $\mu = \mathbb{E}[x]$.

$$\mathrm{Var}(x) \overset{\text{def}}{=} \mathbb{E}\left[\big(x - \mathbb{E}[x]\big)^2\right], \text{ or more generally,}$$

$$\mathrm{Var}(f(x)) \overset{\text{def}}{=} \mathbb{E}\left[\big(f(x) - \mathbb{E}[f(x)]\big)^2\right].$$

It is easy to see that

$$\mathrm{Var}(x) = \mathbb{E}\left[x^2 - 2x\mathbb{E}[x] + \big(\mathbb{E}[x]\big)^2\right] = \mathbb{E}\left[x^2\right] - \big(\mathbb{E}[x]\big)^2,$$

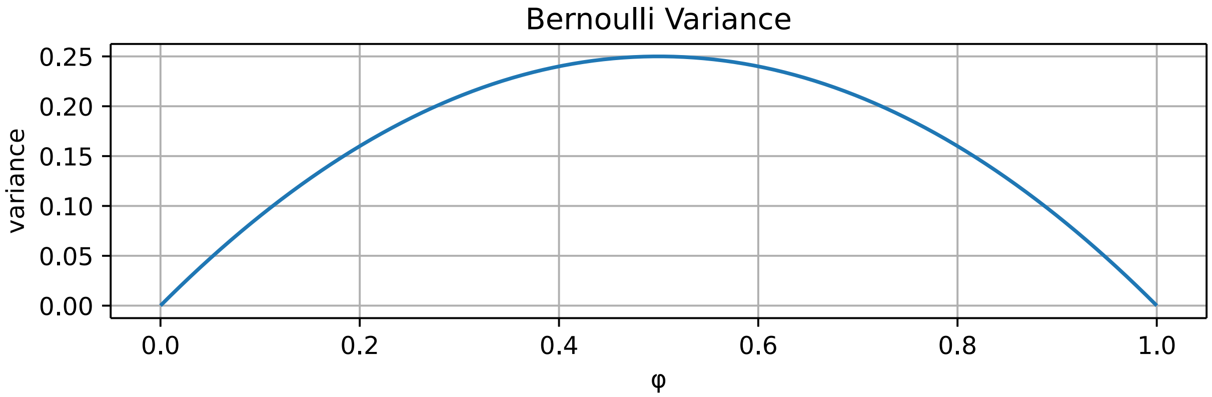
because $\mathbb{E}\big[2x\mathbb{E}[x]\big] = 2(\mathbb{E}[x])^2$.

Variance is connected to $\mathbb{E}[x^2]$, the **second moment** of a random variable – it is in fact a **centered** second moment.

Bernoulli Distribution

The Bernoulli distribution is a distribution over a binary random variable. It has a single parameter $\varphi \in [0, 1]$, which specifies the probability of the random variable being equal to 1.

$$P(x) = \varphi^x (1 - \varphi)^{1-x}$$
$$\mathbb{E}[x] = \varphi$$
$$\mathrm{Var}(x) = \varphi(1 - \varphi)$$



Categorical Distribution

Extension of the Bernoulli distribution to random variables taking one of K different discrete outcomes. It is parametrized by $\boldsymbol{p} \in [0,1]^K$ such that $\sum_{i=0}^{K-1} p_i = 1$.

We represent outcomes as vectors $\in \{0,1\}^K$ in the **one-hot encoding**. Therefore, an outcome $x \in \{0, 1, \ldots, K-1\}$ is represented as a vector

$$\mathbf{1}_x \overset{\text{def}}{=} \big([i = x]\big)_{i=0}^{K-1} = \big(\underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{K-k-1}\big).$$

The outcome probability, mean and variance are very similar to the Bernoulli distribution.

$$P(\boldsymbol{x}) = \prod\nolimits_{i=0}^{K-1} p_i^{x_i}$$

$$\mathbb{E}[x_i] = p_i$$

$$\mathrm{Var}(x_i) = p_i(1 - p_i)$$

Self Information

Amount of **surprise** when a random variable is sampled.

- Should be zero for events with probability 1.
- Less likely events are more surprising.
- Independent events should have **additive** information.

$$I(x) \stackrel{\text{def}}{=} -\log P(x) = \log \frac{1}{P(x)}$$

Entropy

Amount of **surprise** in the whole distribution.

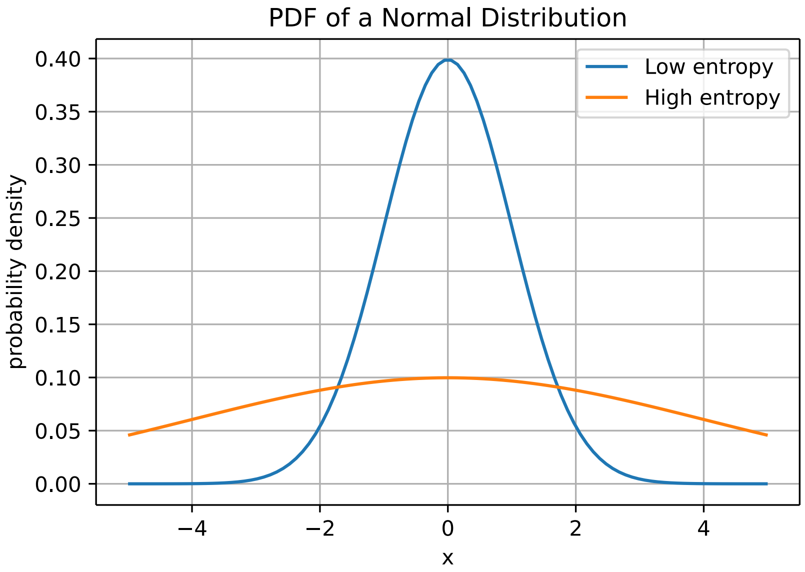
$$H(P) \stackrel{\text{def}}{=} \mathbb{E}_{\mathrm{x}\sim P}[I(x)] = -\mathbb{E}_{\mathrm{x}\sim P}[\log P(x)]$$

- for discrete P: $H(P) = -\sum_x P(x) \log P(x)$
- for continuous P: $H(P) = -\int P(x) \log P(x)\,\mathrm{d}x$

Because $\lim_{x\to 0} x \log x = 0$, for $P(x) = 0$ we consider $P(x) \log P(x)$ to be zero.

Note that in the continuous case, the continuous entropy (also called *differential entropy*) has slightly different semantics, for example, it can be negative.

From now on, all logarithms are *natural logarithms* with base e.



Cross-Entropy

$$H(P, Q) \overset{\text{def}}{=} -\mathbb{E}_{\mathrm{x} \sim P}[\log Q(x)]$$

Gibbs inequality states that

- $H(P, Q) \geq H(P)$
- $H(P) = H(P, Q) \Leftrightarrow P = Q$
- Proof: Using the fact that $\log x \leq (x-1)$ with equality only for $x = 1$, we get

$$\sum_x P(x) \log \frac{Q(x)}{P(x)} \leq \sum_x P(x) \left(\frac{Q(x)}{P(x)} - 1\right) = \sum_x Q(x) - \sum_x P(x) = 0.$$

- Corollary: For a categorical distribution with n outcomes, $H(P) \leq \log n$, because for $Q(x) = 1/n$ we get $H(P) \leq H(P, Q) = -\sum_x P(x) \log Q(x) = \log n$.

Note that generally $H(P, Q) \neq H(Q, P)$.

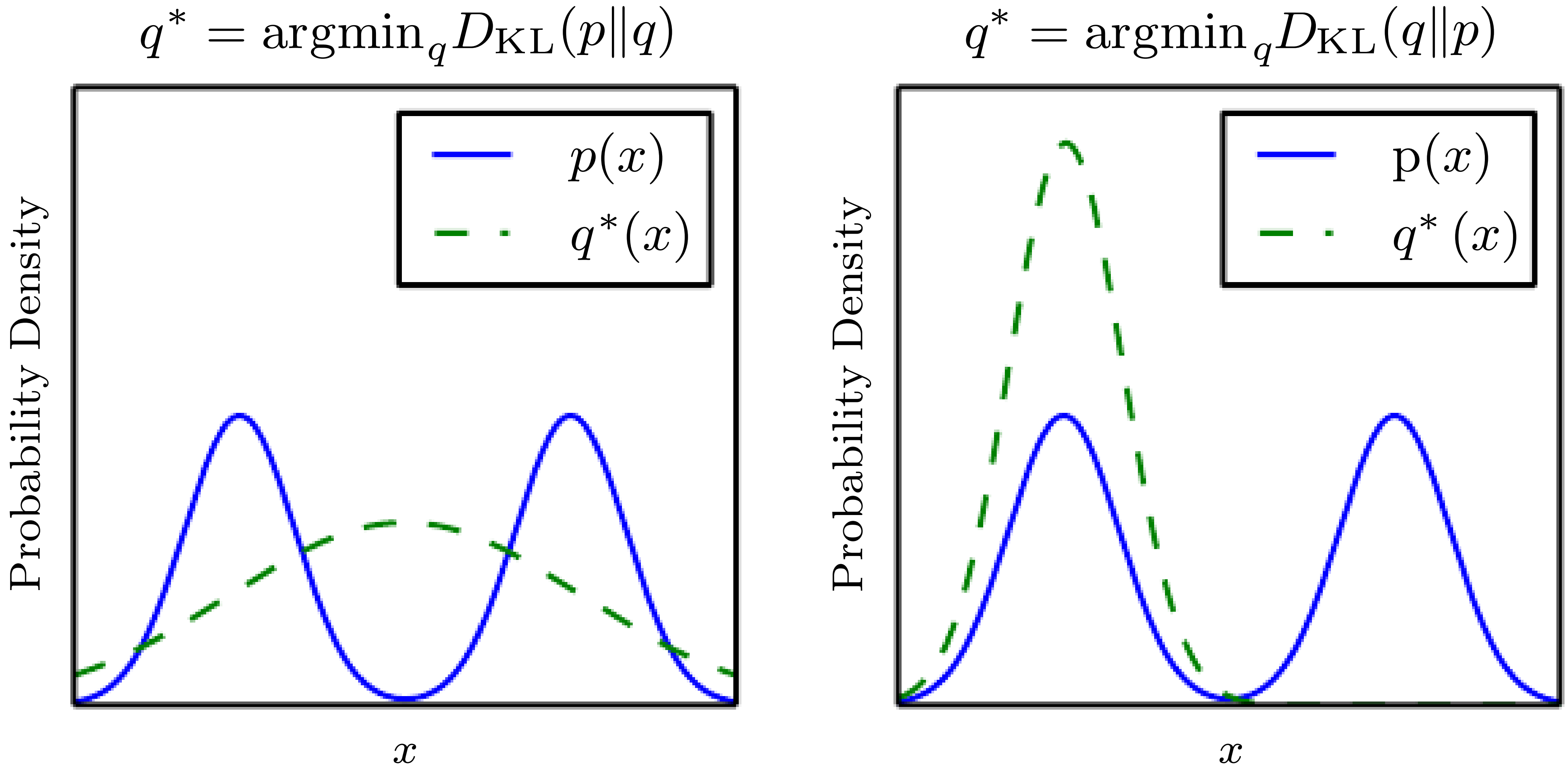
Kullback-Leibler Divergence (KL Divergence)

Sometimes also called **relative entropy**.

$$D_{\mathrm{KL}}(P\|Q) \overset{\text{def}}{=} H(P,Q) - H(P) = \mathbb{E}_{\mathrm{x}\sim P}\big[\log P(x) - \log Q(x)\big]$$

- consequence of Gibbs inequality: $D_{\mathrm{KL}}(P\|Q) \geq 0$, $D_{\mathrm{KL}}(P\|Q) = 0$ iff $P = Q$
- generally $D_{\mathrm{KL}}(P\|Q) \neq D_{\mathrm{KL}}(Q\|P)$

Nonsymmetry of KL Divergence



$q^* = \operatorname{argmin}_q D_{\mathrm{KL}}(p\|q)$

$q^* = \operatorname{argmin}_q D_{\mathrm{KL}}(q\|p)$

Figure 3.6 of "Deep Learning" book, https://www.deeplearningbook.org

Normal (or Gaussian) Distribution

Distribution over real numbers, parametrized by a mean μ and variance σ^2:

$$\mathcal{N}(x;\mu,\sigma^2) = \sqrt{\frac{1}{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

For standard values $\mu = 0$ and $\sigma^2 = 1$ we get $\mathcal{N}(x;0,1) = \sqrt{\frac{1}{2\pi}}e^{-\frac{x^2}{2}}$.

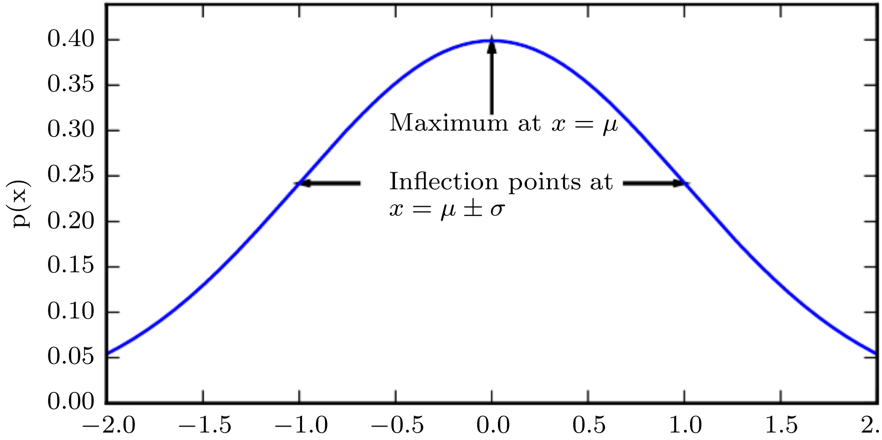


Figure 3.1 of "Deep Learning" book, https://www.deeplearningbook.org

Why Normal Distribution

Central Limit Theorem

The sum of independent identically distributed random variables with finite variance converges to normal distribution.

Principle of Maximum Entropy

Given a set of constraints, a distribution with maximal entropy fulfilling the constraints can be considered the most general one, containing as little additional assumptions as possible.

Considering distributions on all real numbers with a given mean and variance, it can be proven (using variational inference) that such a distribution with **maximum entropy** is exactly the normal distribution.

Machine Learning

A possible definition of learning from Mitchell (1997):

> A computer program is said to learn from experience E with respect to some class of tasks T and performance measure P, if its performance at tasks in T, as measured by P, improves with experience E.

- Task T
  - *classification*: assigning one of k categories to a given input
  - *regression*: producing a number $x \in \mathbb{R}$ for a given input
  - *structured prediction*, *denoising*, *density estimation*, …
- Measure P
  - *accuracy*, *error rate*, *F-score*, …
- Experience E
  - *supervised*: usually a dataset with desired outcomes (*labels* or *targets*)
  - *unsupervised*: usually data without any annotation (raw text, raw images, …)
  - *reinforcement learning*, *semi-supervised learning*, …

Well-known Datasets

Name	Description	Instances
MNIST	Images (28x28, grayscale) of handwritten digits.	60k
CIFAR-10	Images (32x32, color) of 10 classes of objects.	50k
CIFAR-100	Images (32x32, color) of 100 classes of objects (with 20 defined superclasses).	50k
ImageNet	Labeled object image database (labeled objects, some with bounding boxes).	14.2M
ImageNet-ILSVRC	Subset of ImageNet for Large Scale Visual Recognition Challenge, annotated with 1000 object classes and their bounding boxes.	1.2M
COCO	*Common Objects in Context*: Complex everyday scenes with descriptions (5) and highlighting of objects (91 types).	2.5M

Well-known Datasets

ImageNet-ILSVRC

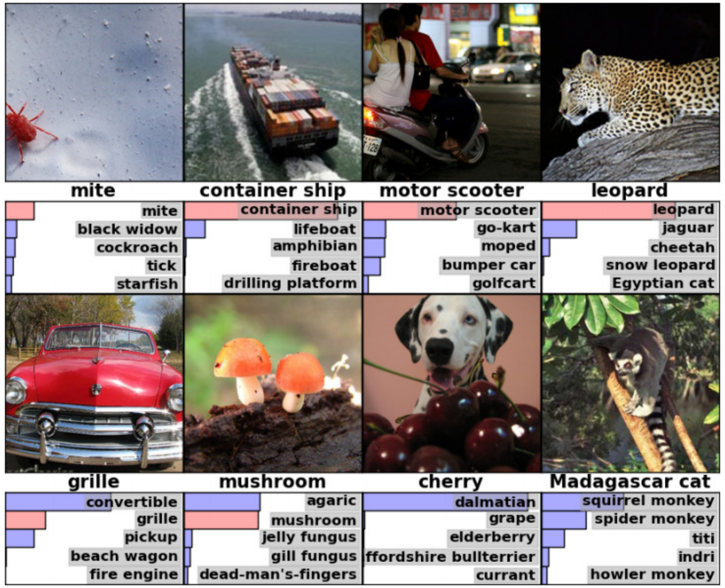
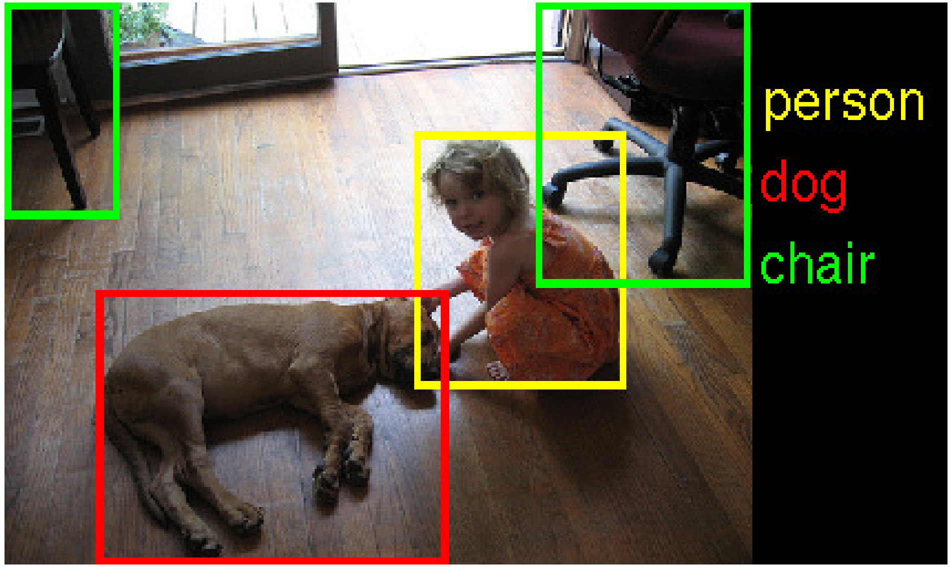


Figure 4 of "ImageNet Classification with Deep Convolutional Neural Networks" by Alex Krizhevsky et al.



https://image-net.org/challenges/LSVRC/2014/

Well-known Datasets

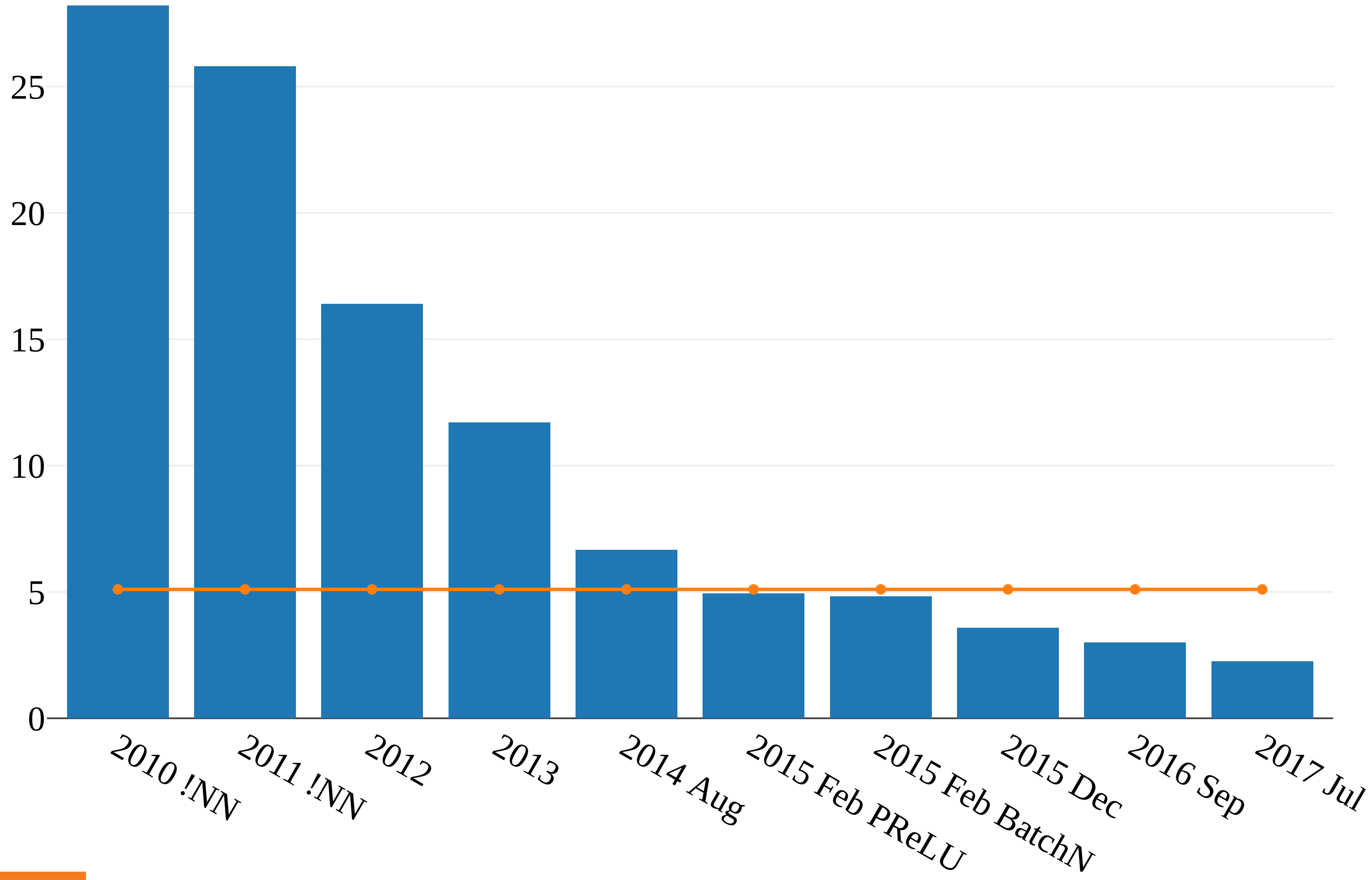
COCO

https://cocodataset.org/#detection-2020

Well-known Datasets

Name	Description	Instances
IAM-OnDB	Pen tip movements of handwritten English from 221 writers.	86k words
TIMIT	Recordings of 630 speakers of 8 dialects of American English.	6.3k sents
CommonVoice	400k recordings from 20k people, around 500 hours of speech.	400k
PTB	*Penn Treebank*: 2500 stories from Wall Street Journal, with POS tags and parsed into trees.	1M words
PDT	*Prague Dependency Treebank*: Czech sentences annotated on 4 layers (word, morphological, analytical, tectogrammatical).	1.9M words
UD	*Universal Dependencies*: Treebanks of 104 languages with consistent annotation of lemmas, POS tags, morphology, syntax.	183 treebanks
WMT	Aligned parallel sentences for machine translation.	gigawords

ILSVRC Image Recognition Error Rates



ILSVRC Image Recognition Error Rates

In summer 2017, a paper came out describing automatic generation of neural architectures using reinforcement learning.

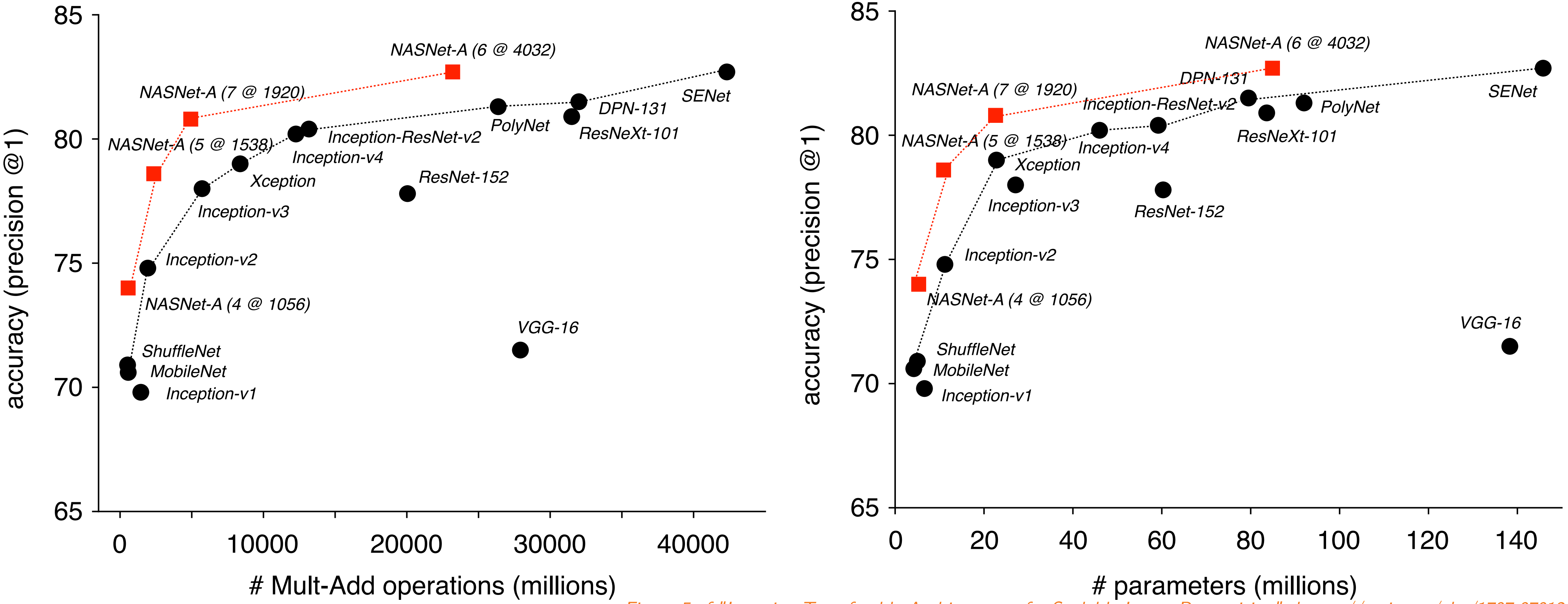
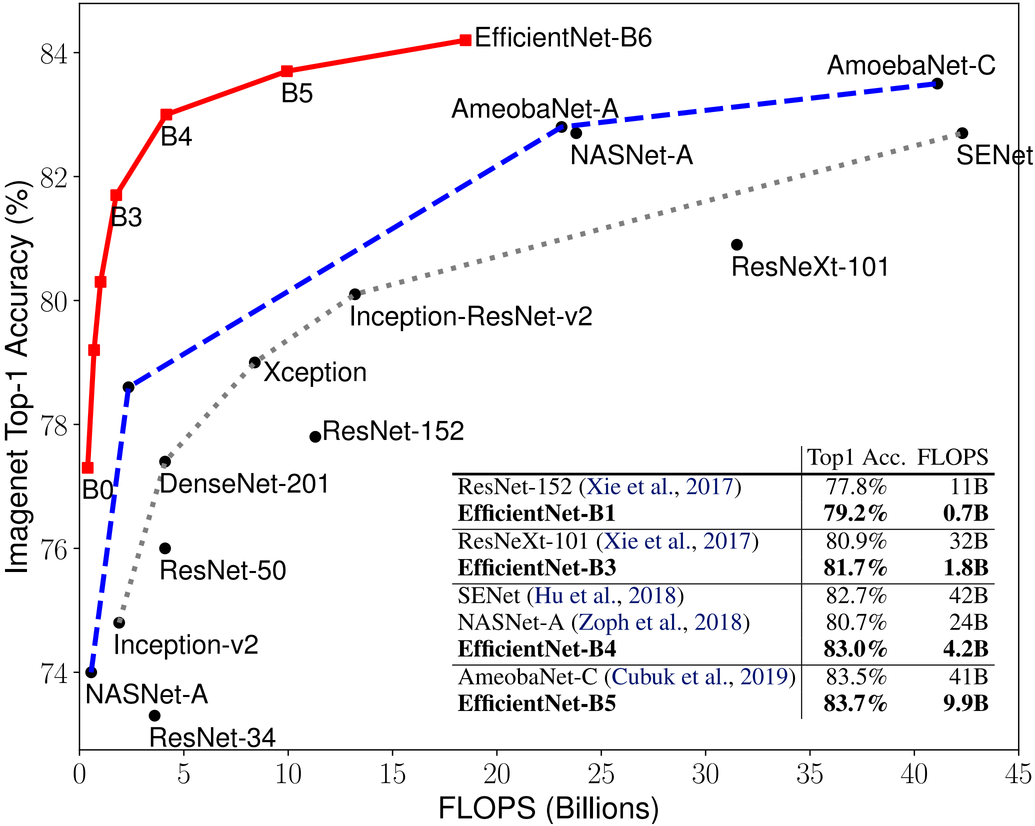


Figure 5 of "Learning Transferable Architectures for Scalable Image Recognition", https://arxiv.org/abs/1707.07012

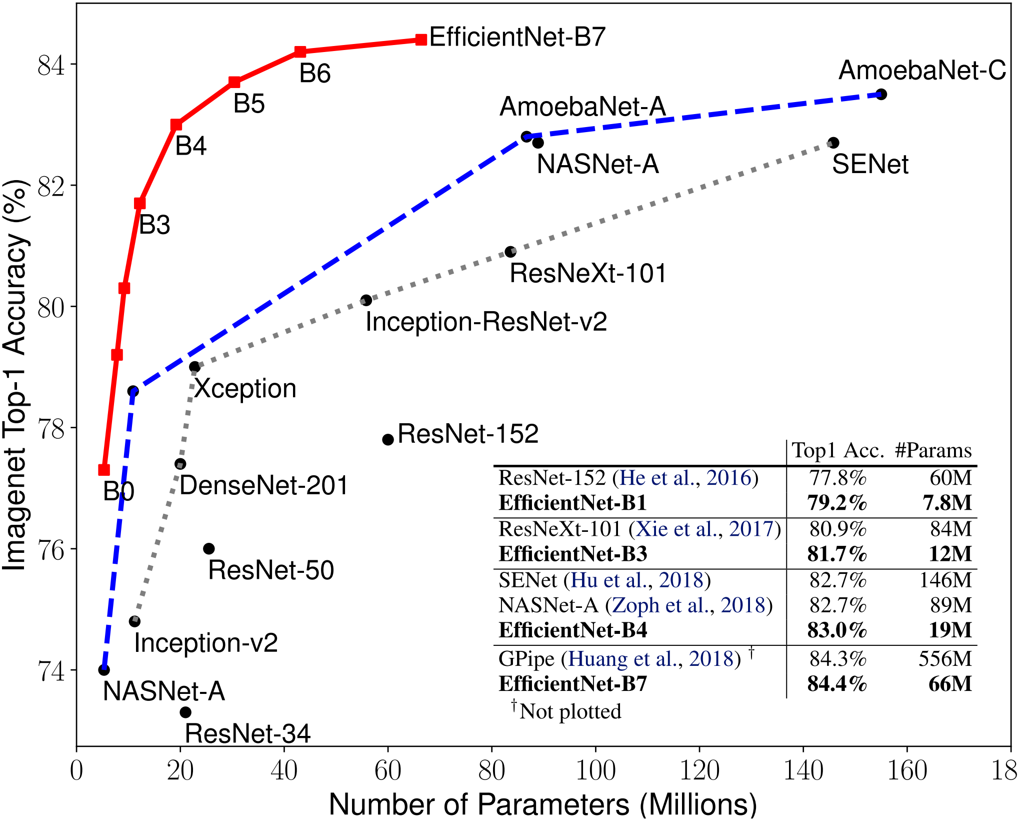
ILSVRC Image Recognition Error Rates

Currently, one of the best architectures is EfficientNet, which combines automatic architecture discovery, multidimensional scaling and elaborate dataset augmentation methods.



	Top1 Acc.	FLOPS
ResNet-152 (Xie et al., 2017)	77.8%	11B
EfficientNet-B1	**79.2%**	**0.7B**
ResNeXt-101 (Xie et al., 2017)	80.9%	32B
EfficientNet-B3	**81.7%**	**1.8B**
SENet (Hu et al., 2018)	82.7%	42B
NASNet-A (Zoph et al., 2018)	80.7%	24B
EfficientNet-B4	**83.0%**	**4.2B**
AmeobaNet-C (Cubuk et al., 2019)	83.5%	41B
EfficientNet-B5	**83.7%**	**9.9B**

Figure 5 of "EfficientNet: Rethinking Model Scaling for Convolutional Neural Networks", https://arxiv.org/abs/1905.11946



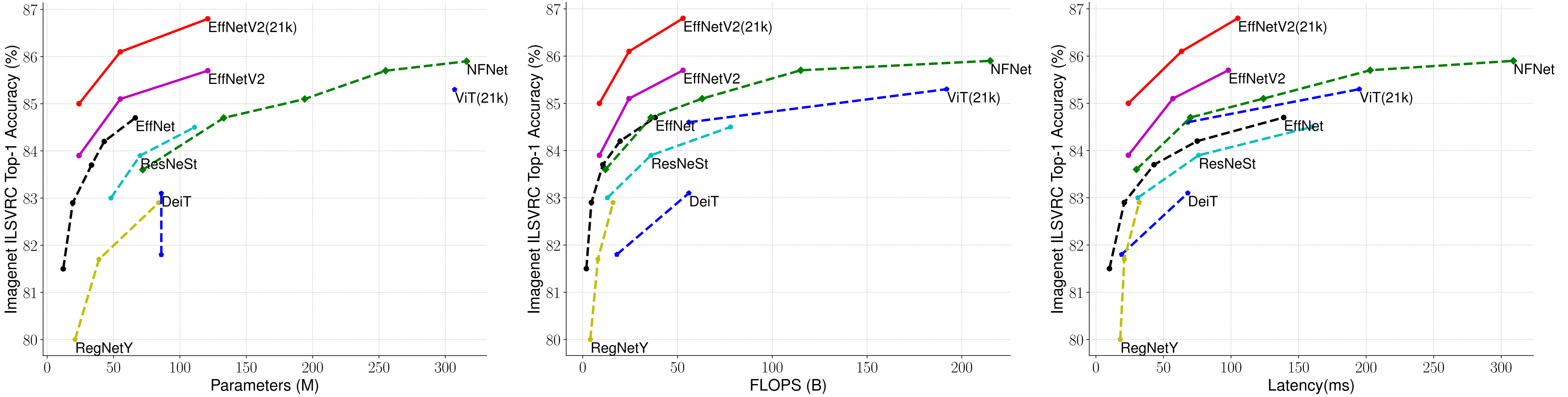
	Top1 Acc.	#Params
ResNet-152 (He et al., 2016)	77.8%	60M
EfficientNet-B1	**79.2%**	**7.8M**
ResNeXt-101 (Xie et al., 2017)	80.9%	84M
EfficientNet-B3	**81.7%**	**12M**
SENet (Hu et al., 2018)	82.7%	146M
NASNet-A (Zoph et al., 2018)	82.7%	89M
EfficientNet-B4	**83.0%**	**19M**
GPipe (Huang et al., 2018) †	84.3%	556M
EfficientNet-B7	**84.4%**	**66M**

†Not plotted

Figure 1 of "EfficientNet: Rethinking Model Scaling for Convolutional Neural Networks", https://arxiv.org/abs/1905.11946

ILSVRC Image Recognition Error Rates

EfficientNet was further improved by EfficientNetV2 two years later.



(a) Parameters

(b) FLOPs

(c) GPU V100 Latency (batch 16)

Figure 5. **Model Size, FLOPs, and Inference Latency** – Latency is measured with batch size 16 on V100 GPU. `21k` denotes pretrained on ImageNet21k images, others are just trained on ImageNet ILSVRC2012. Our EfficientNetV2 has slightly better parameter efficiency with EfficientNet, but runs 3x faster for inference.

Figure 5 of "EfficientNetV2: Smaller Models and Faster Training", https://arxiv.org/abs/2104.00298

Machine Translation Improvements

To illustrate deep neural networks improvements in other domains, consider the English→Czech results of the international Workshop on Machine Translation. Both the automatic BLEU metric and manual evaluation are presented.

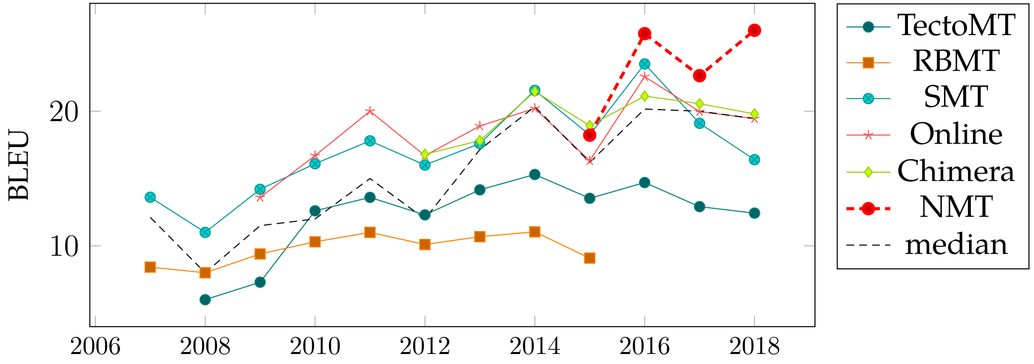


Figure 6.1: WMT English→Czech BLEU evaluation.

Figure 6.1 of "Machine Translation Using Syntactic Analysis", https://dspace.cuni.cz/handle/20.500.11956/104305

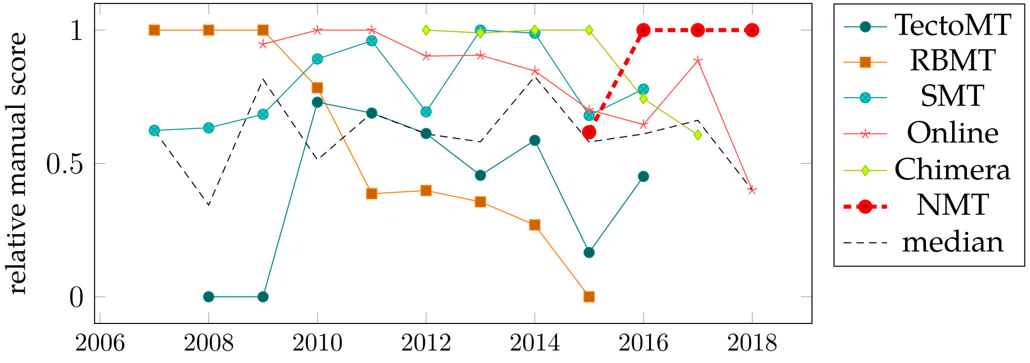
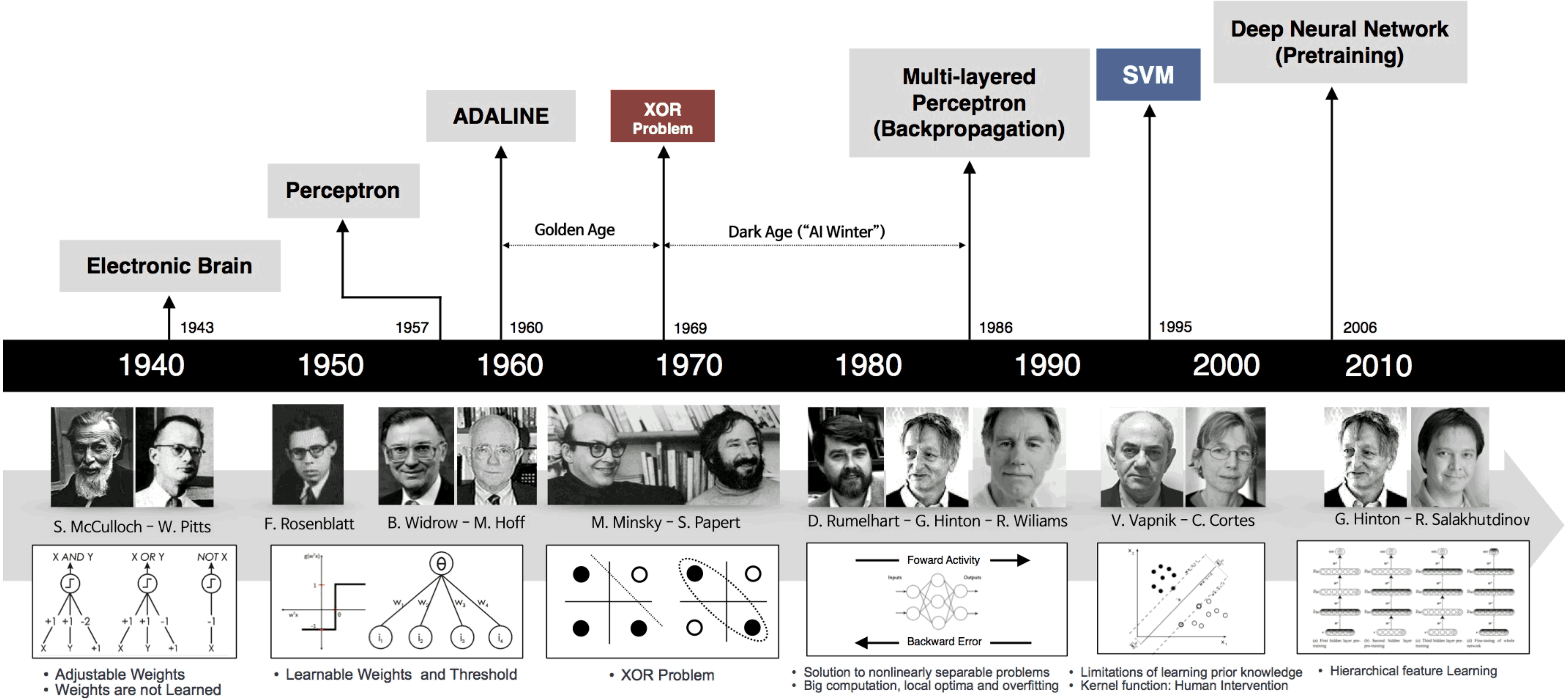


Figure 6.2: WMT English→Czech manual evaluation (higher=better).

Figure 6.2 of "Machine Translation Using Syntactic Analysis", https://dspace.cuni.cz/handle/20.500.11956/104305

- TectoMT parses the input, transfers to the other language, generates the sentence;
- RBMT is the PC-Translator software;
- SMT is statistical machine translation using the Moses system;
- Online is an online translation system (Google in 2009, `Online-B` since 2010);
- **NMT** is the neural machine translation using deep neural networks.

Introduction to Deep Learning History



Electronic Brain — 1943

Perceptron — 1957

ADALINE — 1960

XOR Problem — 1969

Multi-layered Perceptron (Backpropagation) — 1986

SVM — 1995

Deep Neural Network (Pretraining) — 2006

Golden Age

Dark Age (“AI Winter”)

1940 1950 1960 1970 1980 1990 2000 2010

S. McCulloch - W. Pitts

- Adjustable Weights
- Weights are not Learned

F. Rosenblatt

B. Widrow - M. Hoff

- Learnable Weights and Threshold

M. Minsky - S. Papert

- XOR Problem

D. Rumelhart - G. Hinton - R. Wiliams

- Solution to nonlinearly separable problems
- Big computation, local optima and overfitting

V. Vapnik - C. Cortes

- Limitations of learning prior knowledge
- Kernel function: Human Intervention

G. Hinton - R. Salakhutdinov

- Hierarchical feature Learning

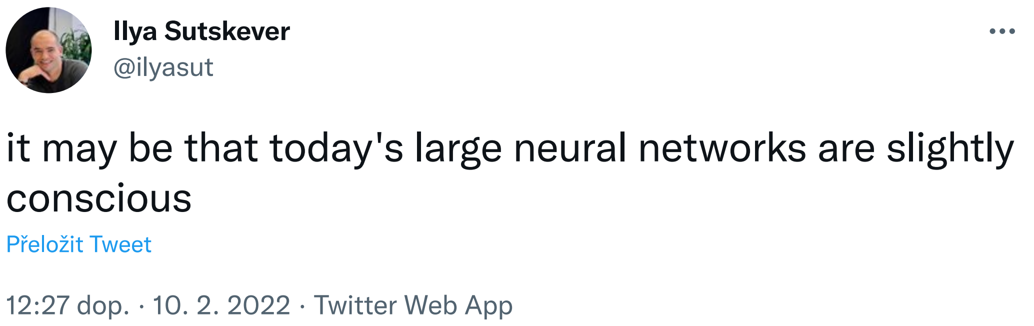
Modified from https://www.slideshare.net/deview/251-implementing-deep-learning-using-cu-dnn/4

How Good is Current Deep Learning

- DL has seen amazing progress in the last ten years.
- Is it enough to get a bigger brain (datasets, models, computer power)?
- Problems compared to Human learning:
  - Sample efficiency
  - Human-provided labels
  - Robustness to data distribution change
  - Stupid errors

https://intl.startrek.com/sites/default/files/styles/content_full/public/images/2019-07/c8ffe9a587b126f152ed3d89a146b445.jpg

How Good is Current Deep Learning



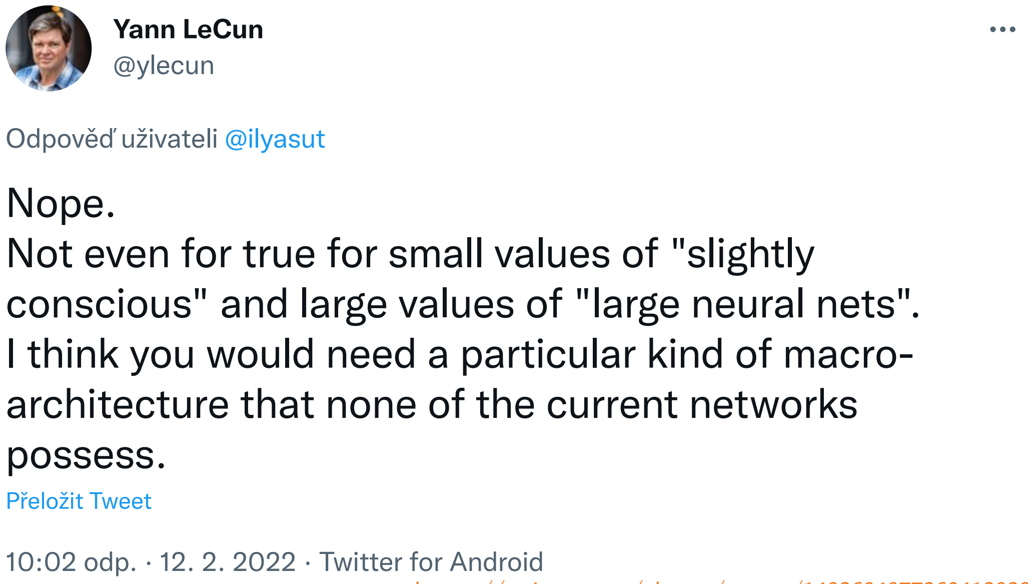
Ilya Sutskever
@ilyasut

it may be that today's large neural networks are slightly conscious

Přeložit Tweet

12:27 dop. · 10. 2. 2022 · Twitter Web App

https://twitter.com/ilyasut/status/1491554478243258368



Yann LeCun
@ylecun

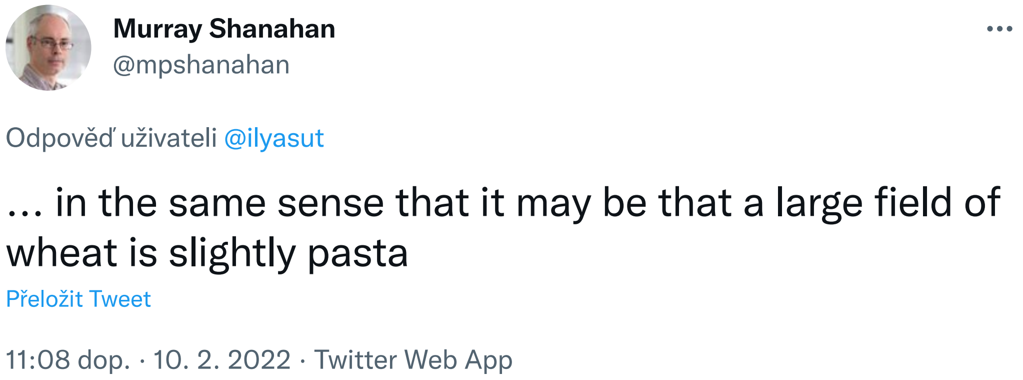
Odpověď uživateli @ilyasut

Nope.
Not even for true for small values of "slightly conscious" and large values of "large neural nets".
I think you would need a particular kind of macro-architecture that none of the current networks possess.

Přeložit Tweet

10:02 odp. · 12. 2. 2022 · Twitter for Android

https://twitter.com/ylecun/status/1492604977260412928



Murray Shanahan
@mpshanahan

Odpověď uživateli @ilyasut

… in the same sense that it may be that a large field of wheat is slightly pasta

Přeložit Tweet

11:08 dop. · 10. 2. 2022 · Twitter Web App

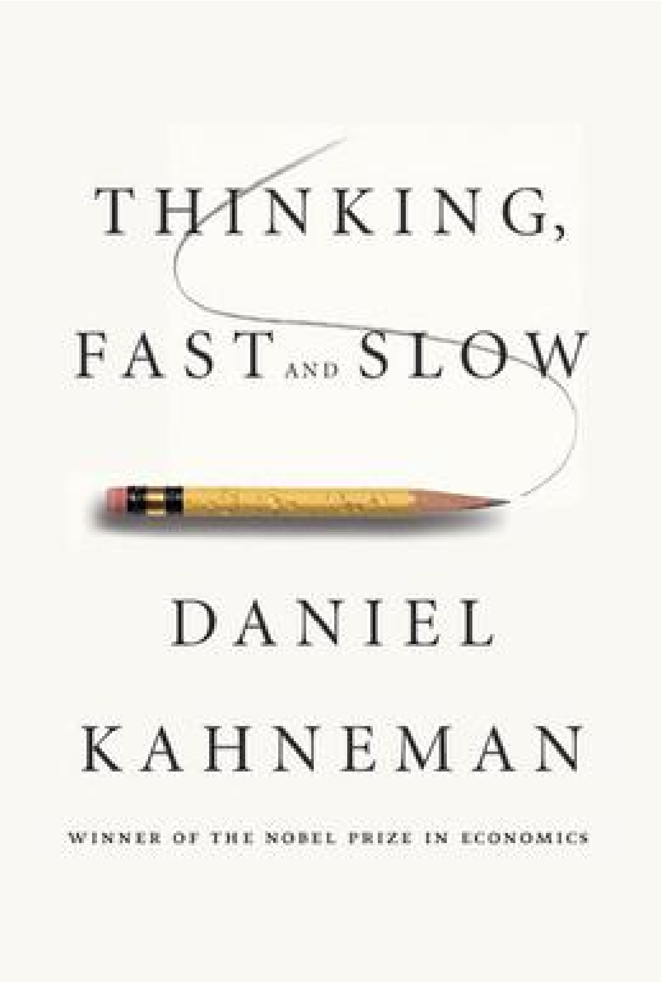
https://twitter.com/mpshanahan/status/1491715721289678848

How Good is Current Deep Learning

- Thinking fast and slow
  - System 1
    - intuitive
    - fast
    - automatic
    - frequent
    - unconscious

    Current DL
  - System 2
    - logical
    - slow
    - effortful
    - infrequent
    - conscious

    Future DL



https://en.wikipedia.org/wiki/File:Thinking,_Fast_and_Slow.jpg

Curse of Dimensionality

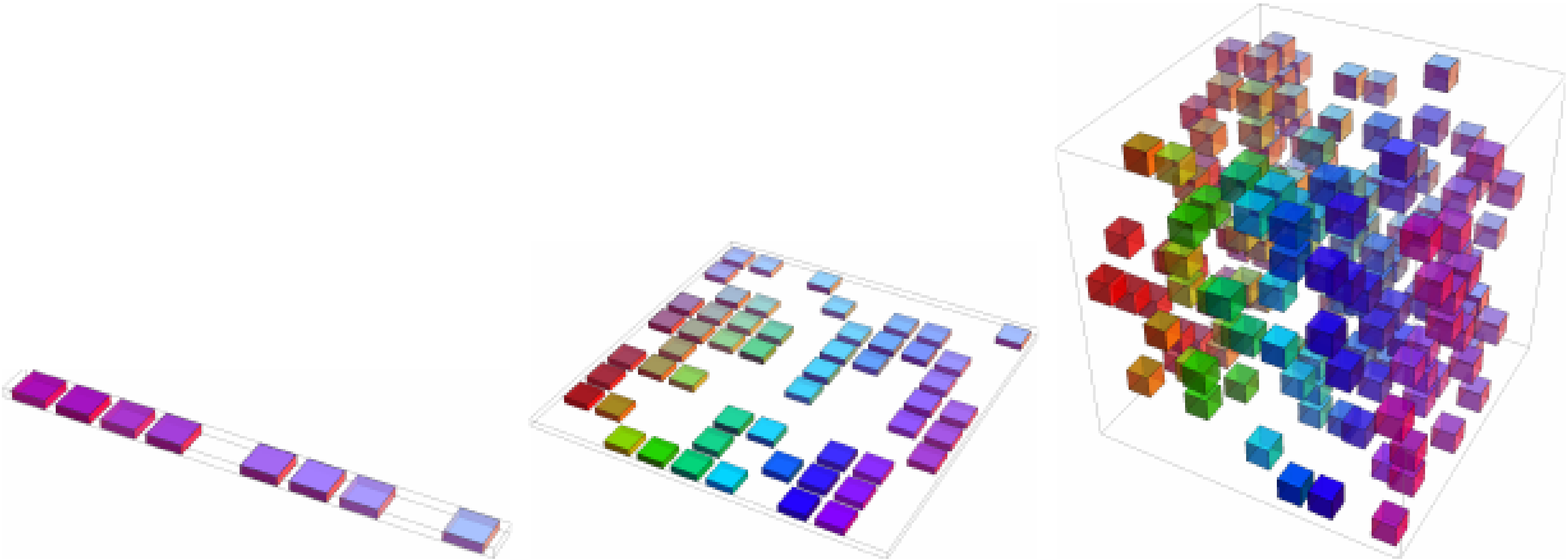


Figure 5.9 of "Deep Learning" book, https://www.deeplearningbook.org

Machine and Representation Learning

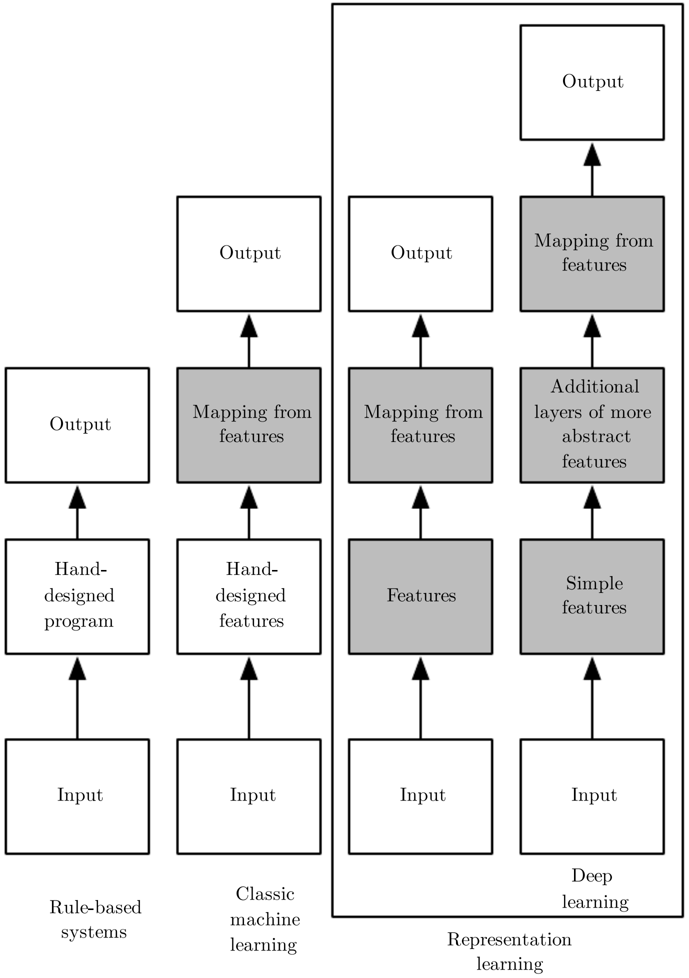
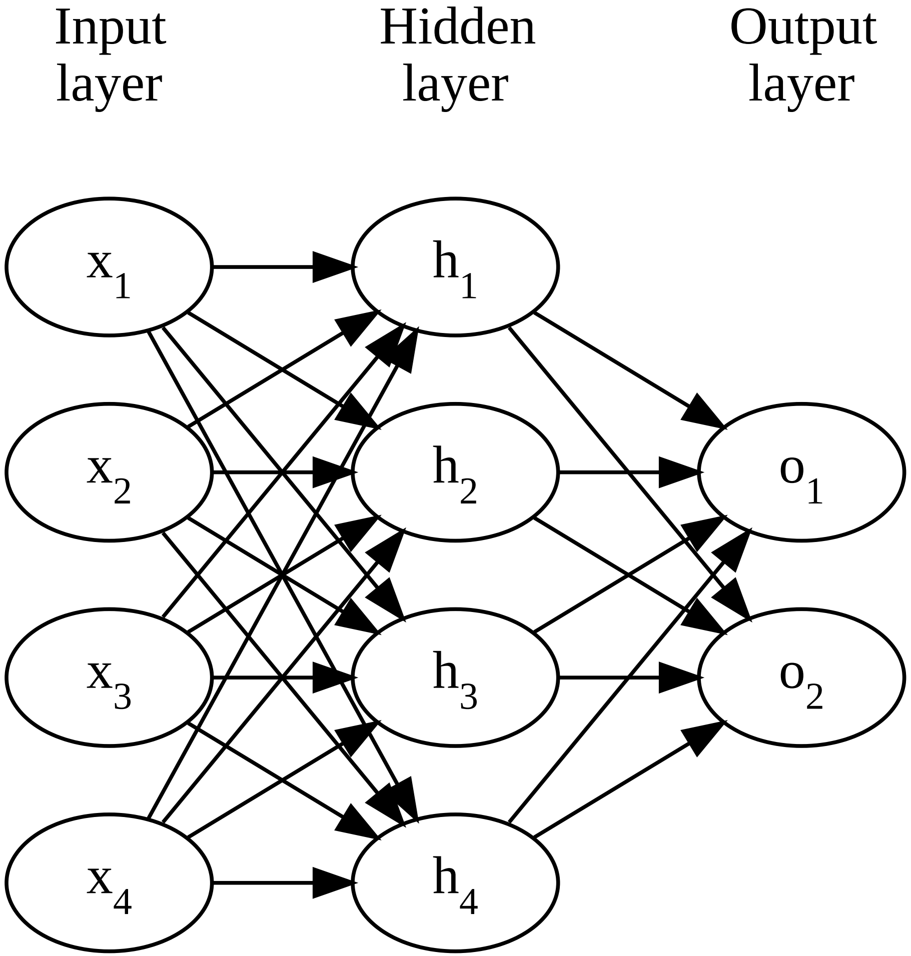


Figure 1.5 of "Deep Learning" book, https://www.deeplearningbook.org

Neural Network Architecture à la '80s



Neural Network Architecture

There is a weight on each edge, and an activation function f is performed on the hidden layers, and optionally also on the output layer.

$$h_i = f\left(\sum_j w_{i,j} x_j + b_i\right)$$

If the network is composed of layers, we can use matrix notation and write

$$\boldsymbol{h} = f\left(\boldsymbol{W}\boldsymbol{x} + \boldsymbol{b}\right),$$

where $\boldsymbol{W} \in \mathbb{R}^{|hidden\ neurons| \times |input\ neurons|}$ is a matrix of weights and $\boldsymbol{b} \in \mathbb{R}^{|hidden\ neurons|}$ is a vector of biases.

Output Layers

- none (linear regression if there are no hidden layers)
- σ (sigmoid; logistic regression if there are no hidden layers)

$$\sigma(x) \stackrel{\text{def}}{=} \frac{1}{1 + e^{-x}}$$

is used to model a probability p of a binary event; its input is called a **logit**, $\log \frac{p}{1-p}$

- $\operatorname{softmax}$ (maximum entropy model if there are no hidden layers)

$$\operatorname{softmax}(\boldsymbol{x}) \propto e^{\boldsymbol{x}}$$

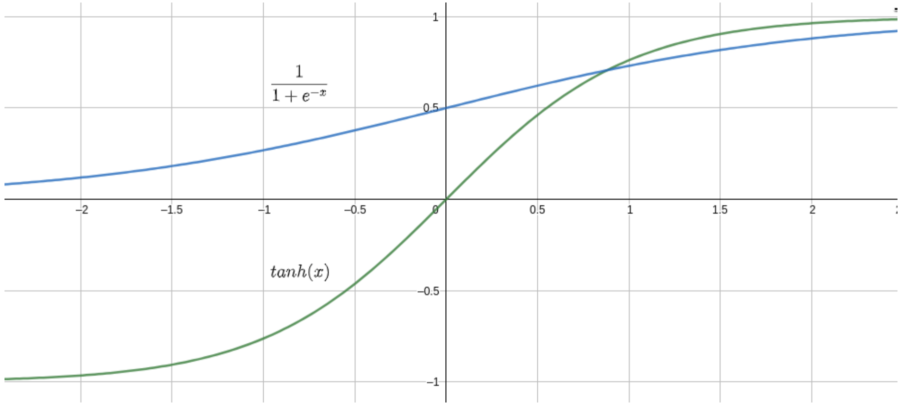
$$\operatorname{softmax}(\boldsymbol{x})_i \stackrel{\text{def}}{=} \frac{e^{x_i}}{\sum_j e^{x_j}}$$

is used to model probability distribution $\boldsymbol{p}$; its input is called a **logit**, $\log(\boldsymbol{p}) + c$

Neural Network Activation Functions

Hidden Layers

- none: does not help, composition of linear mapping is a linear mapping
- σ: however, it works badly – nonsymmetrical, repeated application converges to the fixed point $x = \sigma(x) \approx 0.659$, and $\frac{d\sigma}{dx}(0) = 1/4$
- $\tanh$
  - result of making σ symmetrical and making the derivative in zero 1
  - $\tanh(x) = 2\sigma(2x) - 1$



- ReLU: $\max(0, x)$

Universal Approximation Theorem '89

Let $\varphi(x): \mathbb{R} \to \mathbb{R}$ be a nonconstant, bounded and nondecreasing continuous function. (Later a proof was given also for $\varphi = \mathrm{ReLU}$ and even for any nonpolynomial function.)

For any $\varepsilon > 0$ and any continuous function $f: [0,1]^D \to \mathbb{R}$, there exists $H \in \mathbb{N}$, $\boldsymbol{v} \in \mathbb{R}^H$, $\boldsymbol{b} \in \mathbb{R}^H$ and $\boldsymbol{W} \in \mathbb{R}^{H\times D}$, such that if we denote

$$F(\boldsymbol{x}) = \boldsymbol{v}^T\varphi(\boldsymbol{W}\boldsymbol{x} + \boldsymbol{b}) = \sum_{i=1}^H v_i\varphi(\boldsymbol{W}_i^T\boldsymbol{x} + b_i),$$

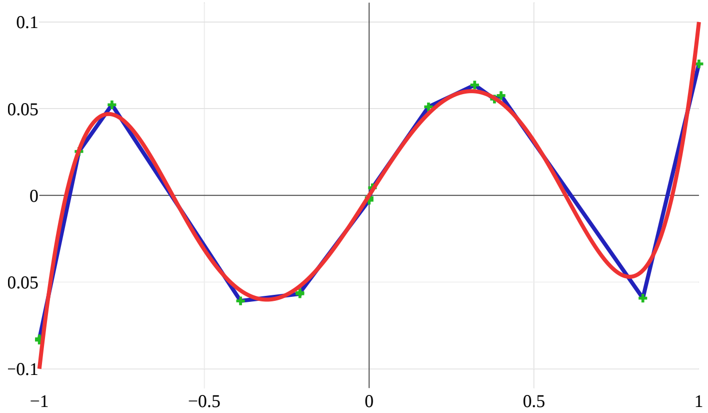
where φ is applied elementwise, then for all $\boldsymbol{x} \in [0,1]^D$:

$$|F(\boldsymbol{x}) - f(\boldsymbol{x})| < \varepsilon.$$

Universal Approximation Theorem for ReLUs

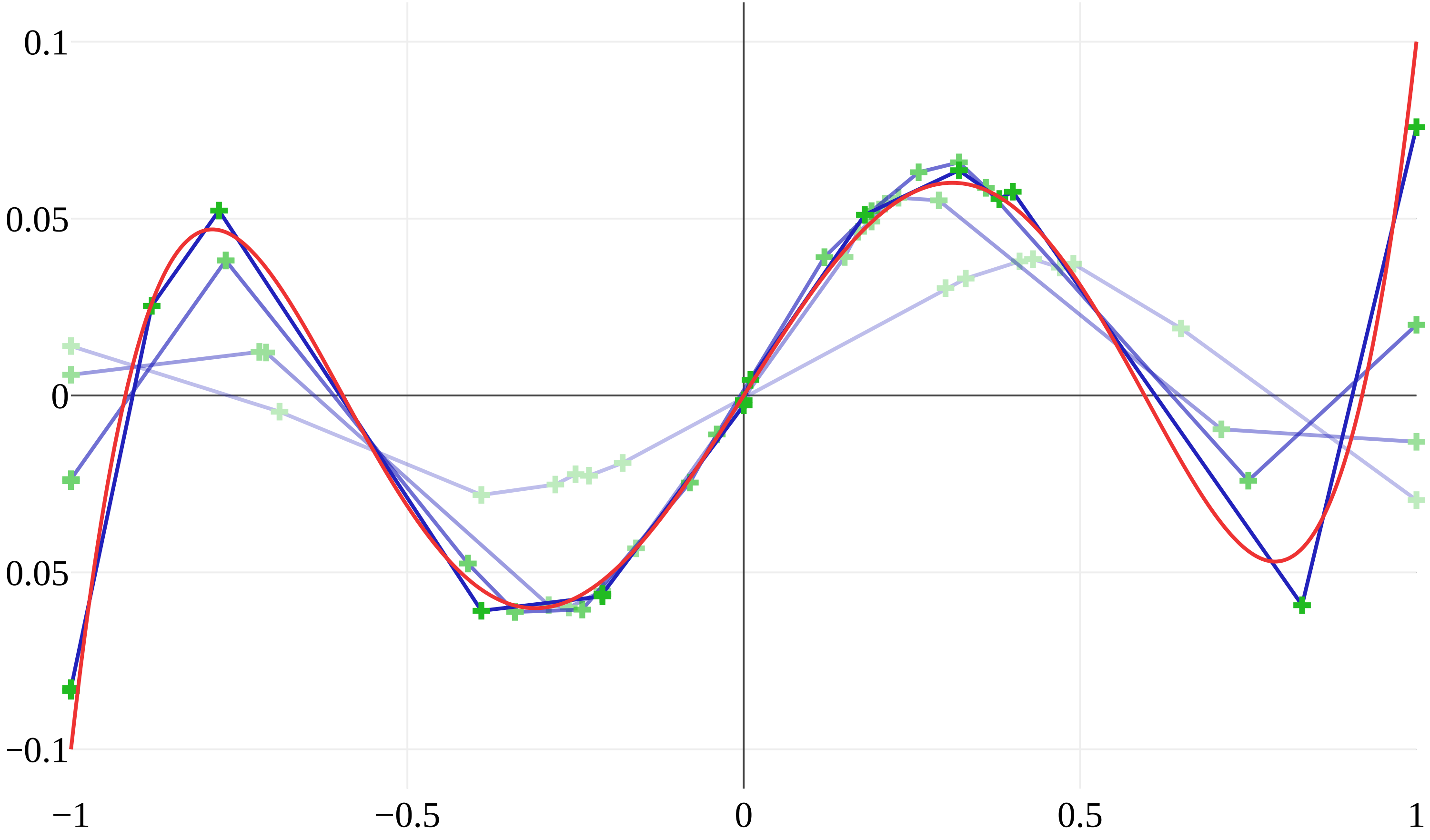
Sketch of the proof:

- If a function is continuous on a closed interval, it can be approximated by a sequence of lines to arbitrary precision.



- However, we can create a sequence of k linear segments as a sum of k ReLU units – on every endpoint a new ReLU starts (i.e., the input ReLU value is zero at the endpoint), with a tangent which is the difference between the target tangent and the tangent of the approximation until this point.

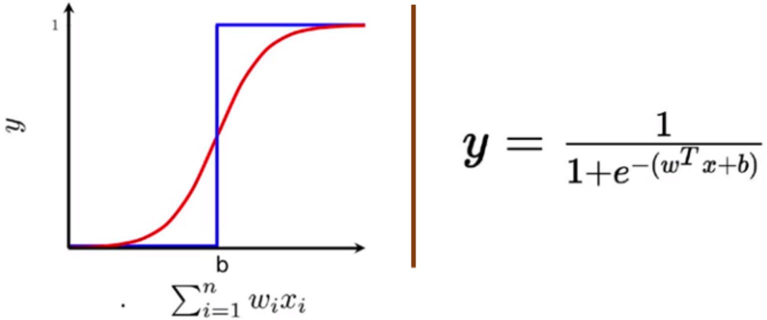
Evolving ReLU Approximation



Universal Approximation Theorem for Squashes

Sketch of the proof for a squashing function $\varphi(x)$ (i.e., nonconstant, bounded and nondecreasing continuous function like sigmoid):

- We can prove φ can be arbitrarily close to a hard threshold by compressing it horizontally.



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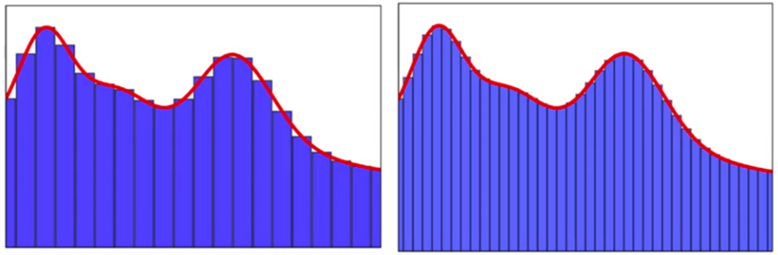
- Then we approximate the original function using a series of straight line segments



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