Statistical Methods in Natural Language Processing

9. Hidden Markov Models and Baum Welch.

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Course Segments

1. Introduction, probability, essential information theory
2. Statistical language modelling (n-gram)
3. Statistical properties of words
4. Word representations
5. Hidden Markov models, Tagging

Recap from Last Week

Markov Properties

- Markov Chain can generalize to any process (not just words):
  - Sequence of random variables: $X = (X_1, X_2, \ldots, X_T)$
  - Sample space S (*states*), size N: $S = \{s_0, s_1, s_2, \ldots, s_N\}$
- Two properties

1. Limited history (context, horizon):

$$\forall i \in 1..T;\ P(X_i | X_1, \ldots, X_{i-1}) = P(X_i | X_{i-1})$$

$\boxed{1\ 7\ 3\ 7\ 9\ 0\ 6\ 7}\ \boxed{3}\ 4\ 5\ldots$ 1 7 3 7 9 0 6 $\boxed{7}\ \boxed{3}\ 4\ 5\ldots$

2. Time invariance (Markov Chain is <u>stationary</u>, <u>homogeneous</u>)

$$\forall i \in 1..T,\ \forall y,x \in S;\ P(X_i = y | X_{i-1} = x) = p(y|x)$$

$1\ \boxed{7}\ \boxed{3}\ \boxed{7}\ \boxed{9}\ 0\ 6\ \boxed{7}\ \boxed{3}\ 4\ 5\ldots$

ok ... same <u>distribution</u>

Long History Possible

- What if we want trigrams:

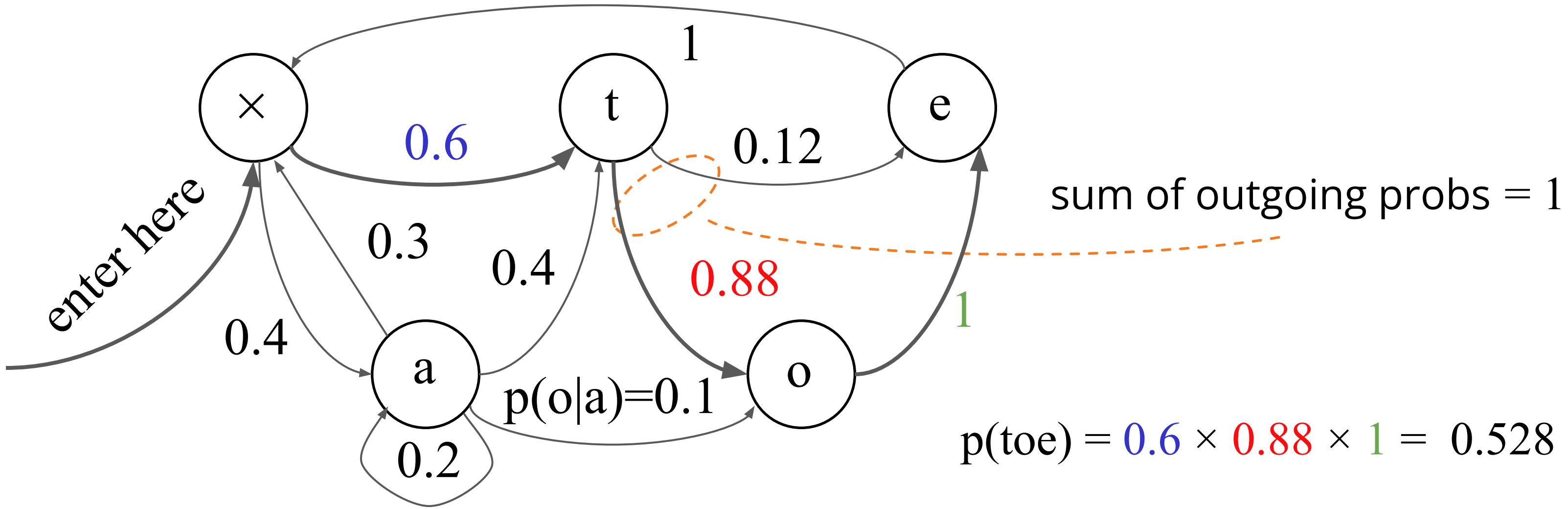
  1 7 3 7 9 0 6 7 3 $\boxed{4\ 5}$ $\boxed{\ldots}$

- Formally, use transformation:
  - Define new variables Q_i, such that $X_i = \{Q_{i-1},Q_i\}$
  - And then $P(X_i|X_{i-1}) = P(Q_{i-1},Q_i|Q_{i-2},Q_{i-1}) = P(Q_i|Q_{i-2},Q_{i-1})$

Predicting (X_i):	1	7	3	7	$\boxed{9\ 0}$	6	7	3	4	5	…
History ($X_{i-1} = \{Q_{i-2},Q_{i-1}\}$):	×	1	7	3	…	0	6	7	3	4	…
	×	×	1	7	…	9	0	6	7	3	…

Markov Models: Bigram Case

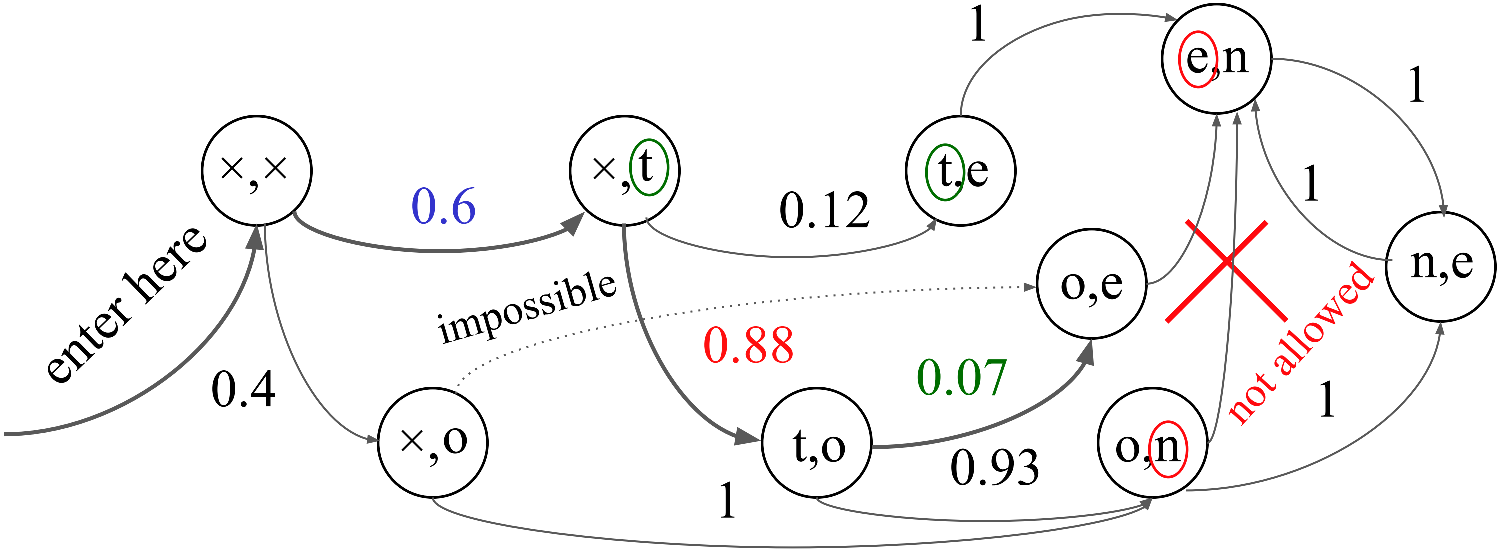
- Nodes: States, $S = \{s_0, s_1, s_2, \ldots, s_N\}$
- Arcs: Transitions with probabilities, $P(X_i|X_{i-1})$, X_i generates s_i



$p(toe) = 0.6 \times 0.88 \times 1 = 0.528$

Markov Models: Trigram Case

- Nodes: Pairs of states (s_k,s_l), $S = \{s_0,s_1,s_2,\ldots,s_N\}$
- Arcs: Transitions with probabilities, $P(X_i|X_{i-1})$, X_i generates (s_k,s_l)



$p(\text{toe}) = 0.6 \times 0.88 \times 0.07 = 0.037$

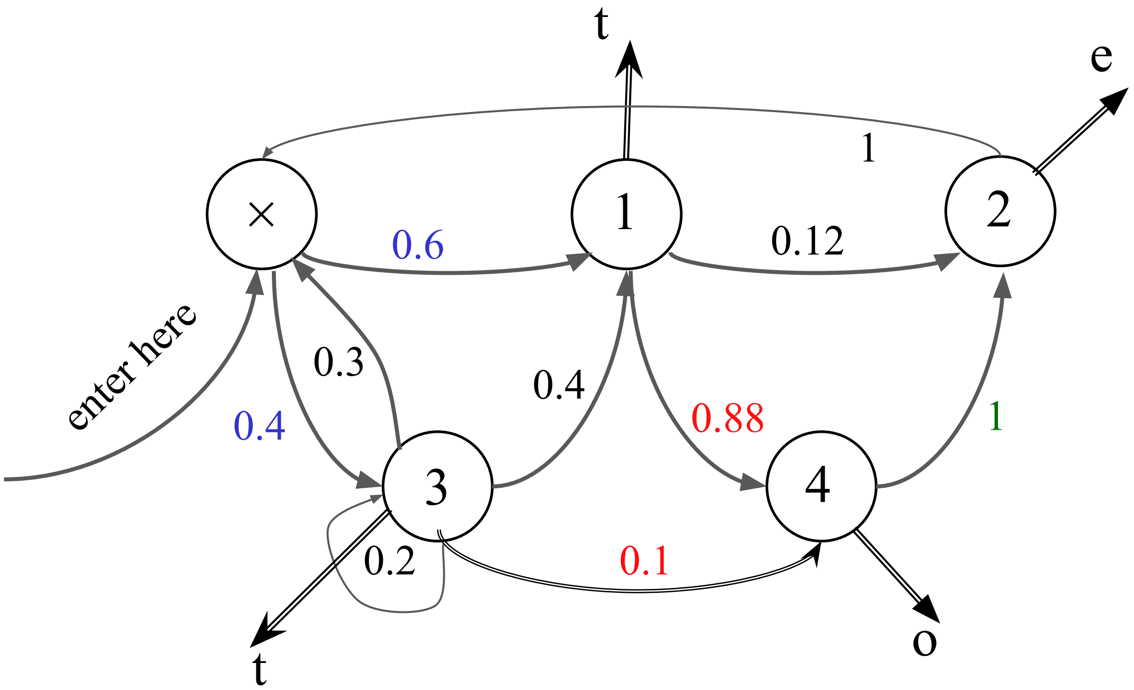
$p(\text{one}) = ?$

Finite State Automaton

- States ~ symbols of the [input / output] alphabet
  - pairs (or more): last element of the n-tuple
- Arcs ~ transitions (sequence of states)
  - Classical FSA: alphabet symbols on arcs:
  - possible transformation: arcs ↔ nodes
- Possible thanks to the “limited history” Markov Property
- So far: ***Visible*** Markov Models (VMM)

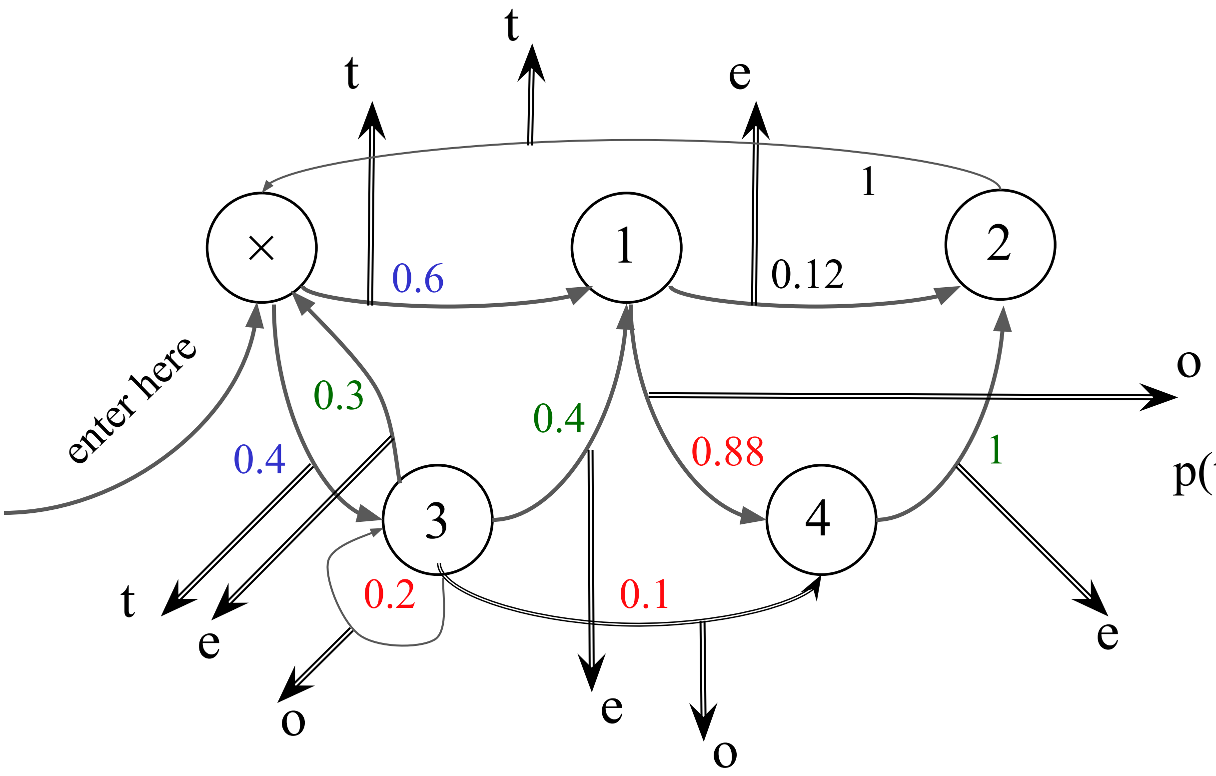
Hidden Markov Models

- The simplest HMM: states generate [*observable*] output (using the "data" alphabet) but remain "*invisible*"



$$p(\text{toe}) = 0.6\times0.88\times1 + 0.4\times0.1\times1 \cong 0.568$$

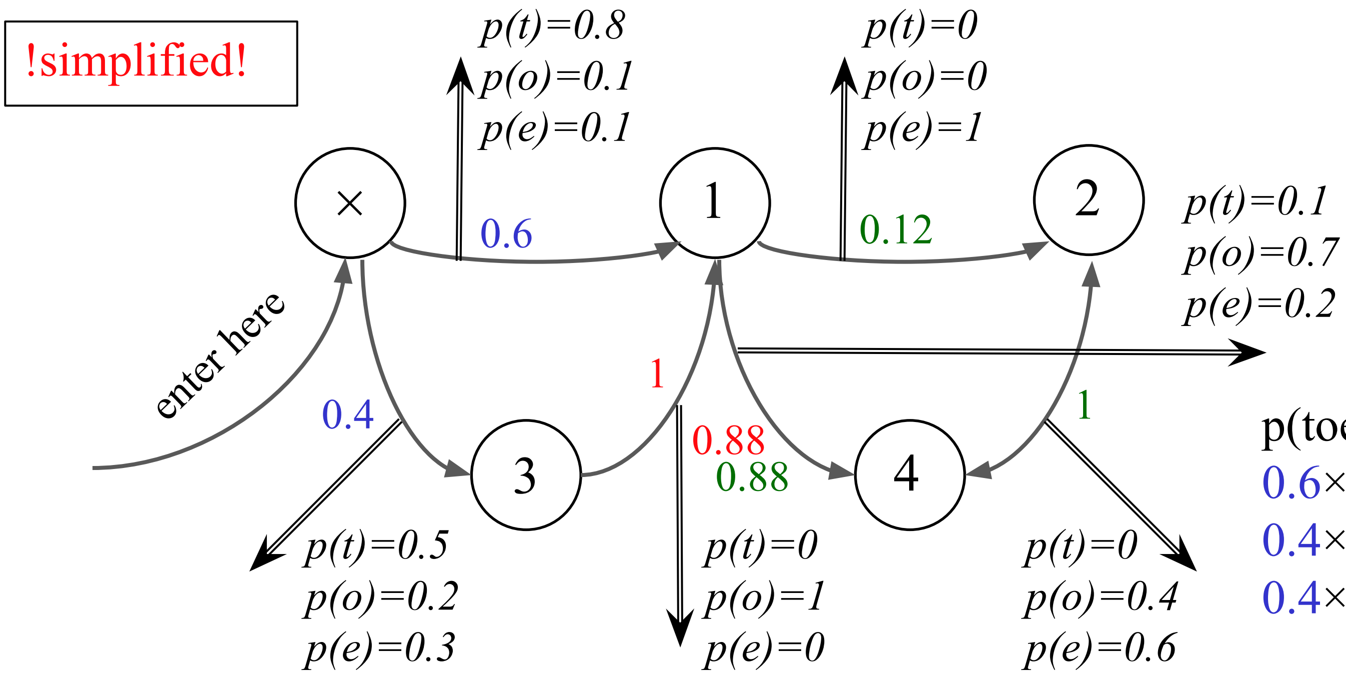
- Added flexibility: Generate output from arcs, not states:



$$
\begin{aligned}
p(toe) = {} & 0.6 \times 0.88 \times 1 \; + \\
& 0.4 \times 0.1 \times 1 \; + \\
& 0.4 \times 0.2 \times 0.3 \; + \\
& 0.4 \times 0.2 \times 0.4 \cong 0.624
\end{aligned}
$$

… and Finally, Add Output Probabilities

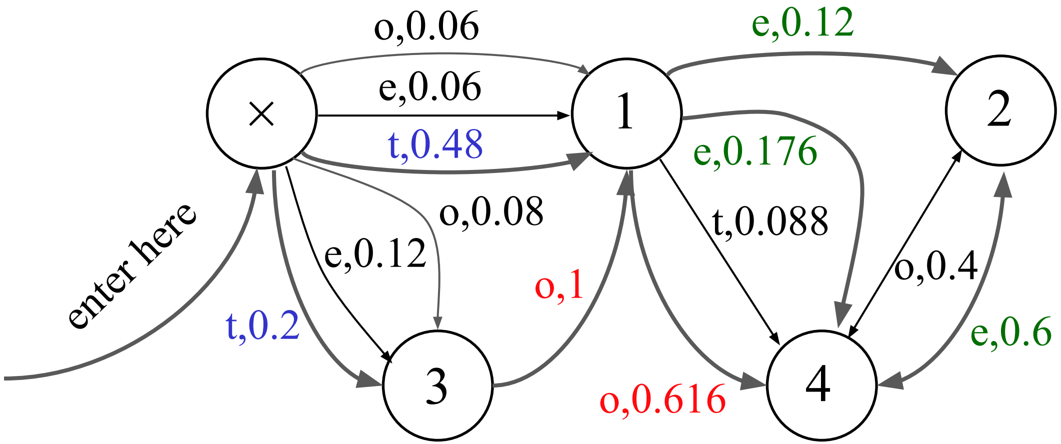
- Maximum flexibility: [Unigram] distribution (sample space: output alphabet) at each output arc:



$$
\begin{aligned}
p(\text{toe}) = {} & 0.6\times0.8\times0.88\times0.7\times1\times0.6+\\
& 0.4\times0.5\times1\times1\times0.88\times0.2 +\\
& 0.4\times0.5\times1\times1\times0.12\times1 \cong 0.237
\end{aligned}
$$

Slightly Different View

- Allow for multiple arcs from $s_i \rightarrow s_j$, mark them by output symbols, get rid of output distributions:



$$p(toe) = 0.48 \times 0.616 \times 0.6 + 0.2 \times 1 \times 0.176 + 0.2 \times 1 \times 0.12 \cong 0.237$$

Formalization

- HMM (the general case): five-tuple $\mathbf{(S, s_0, Y, P_S, P_Y)}$, where:
  - $\mathbf{S = \{s_1, s_2, \ldots, s_T\}}$ is the set of states, $\mathbf{s_0}$ is the initial state,
  - $\mathbf{Y = \{y_1, y_2, \ldots, y_V\}}$ is the output alphabet,
  - $\mathbf{P_S(s_j | s_i)}$ is the set of prob. distributions of transitions,
    - size of P_S: $|S|^2$
  - $\mathbf{P_Y(y_k | s_i, s_j)}$ is the set of output (emission) probability distributions
    - size of P_Y: $|S|^2 \times |Y|$

- Example:
  - $S = \{x, 1, 2, 3, 4\}$, $s_0 = x$
  - $Y = \{ t, o, e \}$

Formalization - Example

- Example (for graph on slide 36):
  - $S = \{x, 1, 2, 3, 4\}$, $s_0 = x$
  - $Y = \{ t, o, e \}$
  - P_S:

	×	**1**	**2**	**3**	**4**
×	0	0.6	0	0.4	0
1	0	0	0.12	0	0.88
2	0	0	0	0	1
3	0	1	0	0	0
4	0	0	1	0	0

$\Sigma = 1$

P_Y:

t	×	**1**	**2**	**3**	**4**
×		0.8		0.5	
1					0.1
2					0
3		0			
4			0		

o	×	**1**	**2**	**3**	**4**
×					
1					0.7

e	×	**1**	**2**	**3**	**4**
×					
1					0.2

$\Sigma = 1$

HMM: The Two Tasks

- HMM (the general case): five-tuple $(\mathbf{S}, \mathbf{S_0}, \mathbf{Y}, \mathbf{P_S}, \mathbf{P_Y})$, where:
  - $\mathbf{S} = \{\mathbf{s_1,s_2,\ldots,s_T}\}$ is the set of states, $\mathbf{S_0}$ is the initial state,
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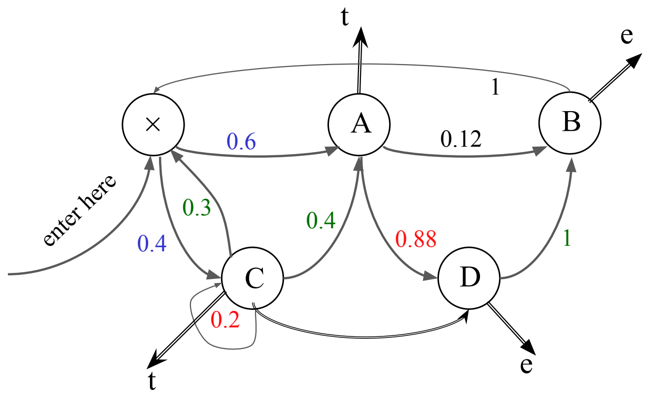
- Given an HMM & an output sequence $Y = \{y_1,y_2,\ldots,y_k\}$:
  - (Task 1) compute the probability of Y;
  - (Task 2) compute the most likely sequence of states which has generated Y.

Trellis

Trellis: HMM “roll-out”

HMM:

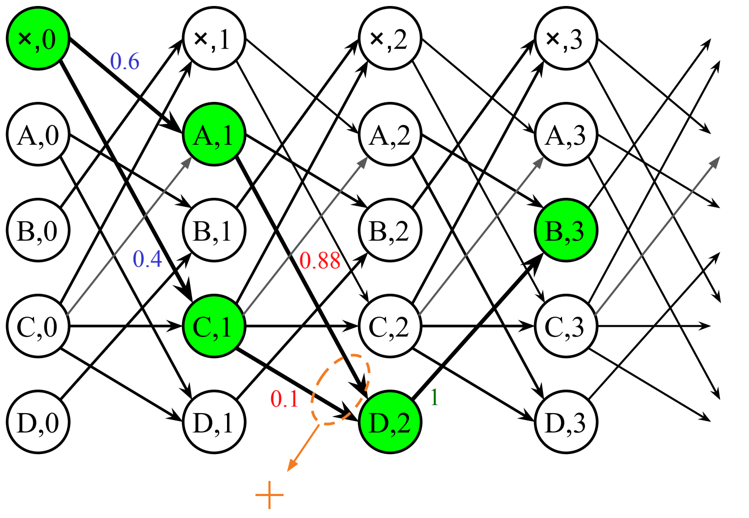


p(toe) = 0.6×0.88×1 + 0.4×1×1 ≅ 0.568

Trellis:

“roll-out”

time/position: 0 1 2 3 ....



Y: t o e

$\alpha(\times,0) = 1$ $\alpha(A,1) = 0.6$ $\alpha(D,2) = 0.568$ $\alpha(B,3) = 0.568$

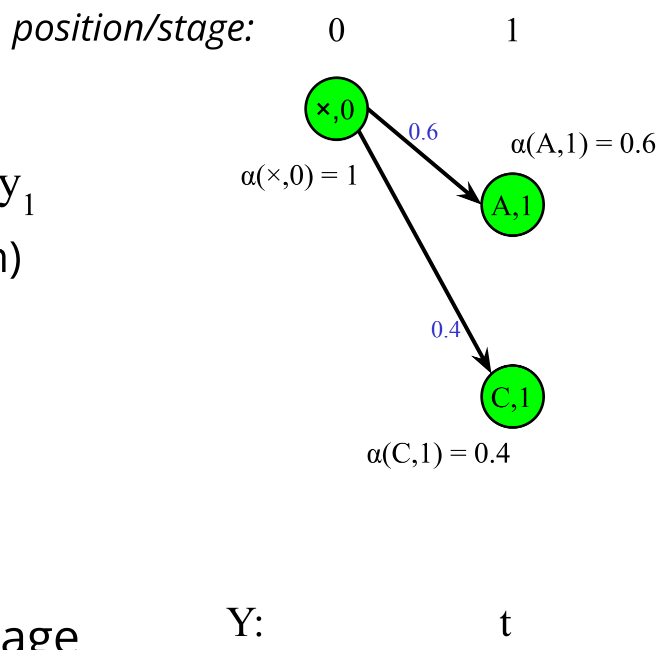
$\alpha(C,1) = 0.4$

- Trellis state: (HMM state, position)
- each state: holds ***one*** number (prob): **α**
- probability of Y: Σα in the last state

Creating the Trellis: The Start

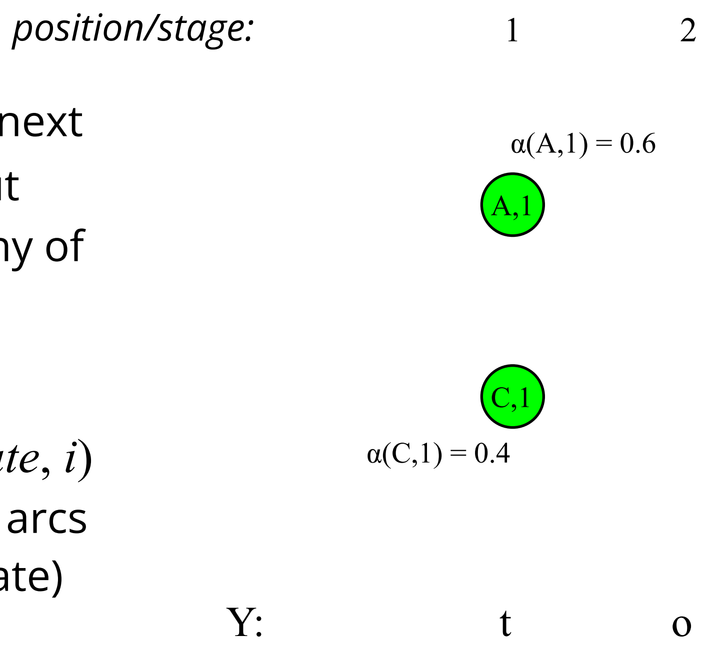
- Start in the start state (×),
  - set its $\alpha(\times,0)$ to 1
- Create the first stage:
  - get the first "output" symbol y_1
  - create the first stage (column)
  - but only those Trellis states

    which generate y_1
  - set their $\alpha(state,1)$ to the $P_S(state|\times)\ \alpha(\times,0)$
- and forget about the *0*-th stage



Trellis: The Next Step

- Suppose we are in stage i
- Creating the next stage:
  - create all trellis states in the next stage which generate y_{i+1}, but only those reachable from any of the stage-i states
  - set their $\alpha(state, i+1)$ to:
    $P_S(state|prev.state) \times \alpha(prev.state, i)$
    (add up all such numbers on arcs going to a common Trellis state)
    ...and forget about stage i



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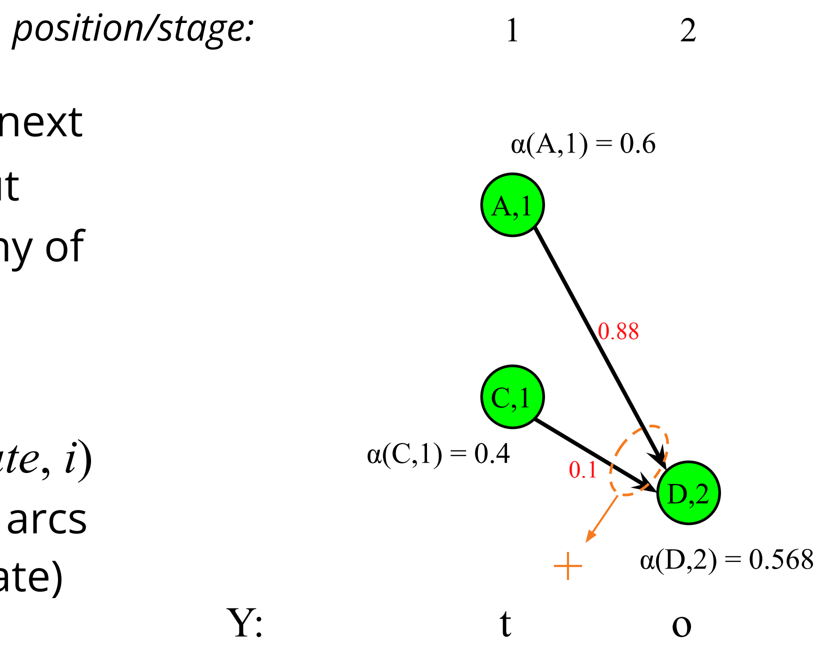
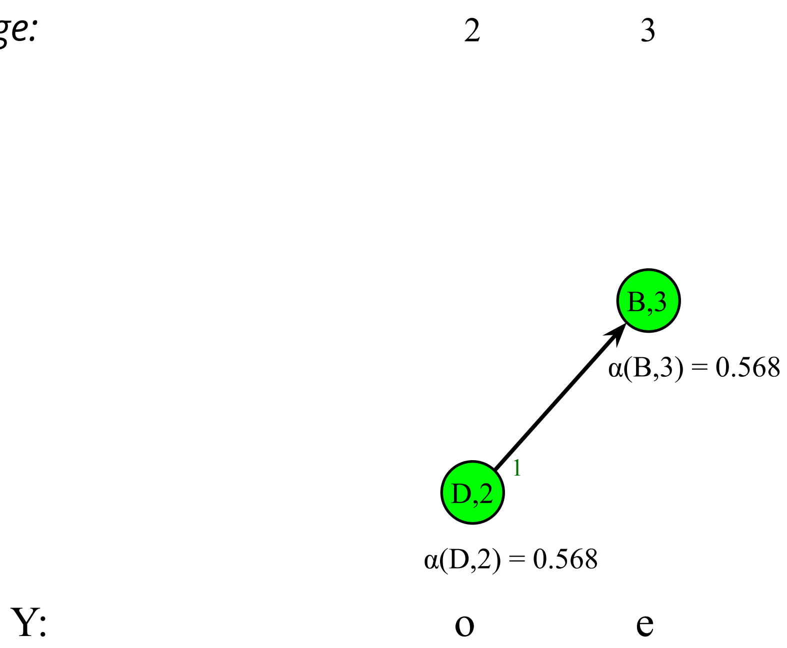
  - set their $\alpha(state,i+1)$ to: $P_S(state|prev.state) \times \alpha(prev.state, i)$ (add up all such numbers on arcs going to a common Trellis state) ...and forget about stage i



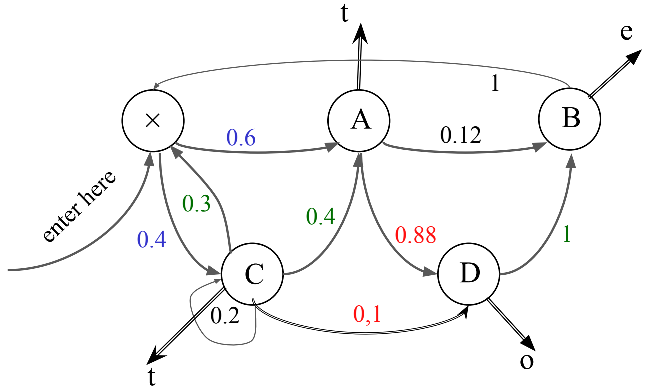
Trellis: The Last Step

- Continue until output exhausted
  - $|Y| = 3$: until stage 3
- Add together all the $\alpha(state, |Y|)$
- That's the <u>P(Y)</u>
- Observation (pleasant):
  - memory usage max: $2|S|$
  - multiplications max: $|S|^2|Y|$



Trellis (again)

HMM:

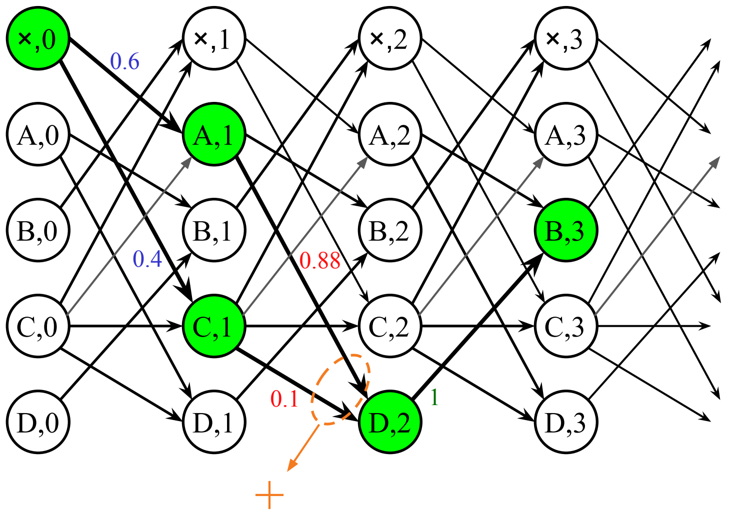


$p(toe) = 0.6 \times 0.88 \times 1 + 0.4 \times 1 \times 1 \cong 0.568$

“roll-out”

Trellis:

time/position: 0 1 2 3 ....



Y: t o e

$\alpha(\times,0) = 1$ $\alpha(A,1) = 0.6$ $\alpha(D,2) = 0.568$ $\alpha(B,3) = 0.568$

$\alpha(C,1) = 0.4$

- Trellis state: (HMM state, position)
- each state: holds ***one*** number (prob): $\boldsymbol{\alpha}$
- probability of Y: $\Sigma\alpha$ in the last state

HMM: The Two Tasks

- HMM (the general case): five-tuple $(\mathbf{S}, \mathbf{S_0}, \mathbf{Y}, \mathbf{P_S}, \mathbf{P_Y})$, where:
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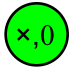
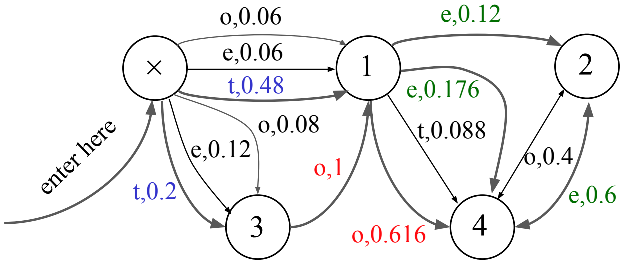
- Given an HMM & an output sequence $\mathrm{Y} = \{\mathrm{y_1,y_2,\ldots,y_k}\}$:
  - (Task 1) compute the probability of Y;
  - (Task 2) compute the most likely sequence of states which has generated Y.

- Start as usual:
    - start state (×), set its $\alpha(\times,0)$ to 1.

position/stage: 0



$\alpha(\times,0) = 1$



$$p(toe) = 0.48 \times 0.616 \times 0.6 + 0.2 \times 1 \times 0.176 + 0.2 \times 1 \times 0.12 \cong 0.237$$

Y:

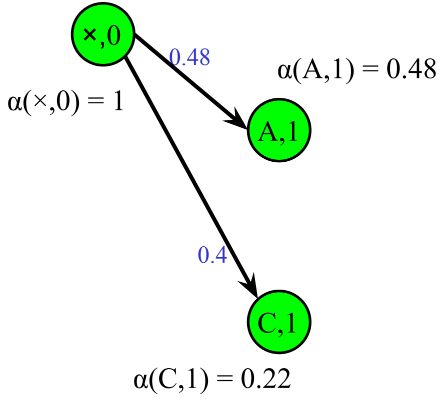
General Trellis: The Next Step

We are in stage i :

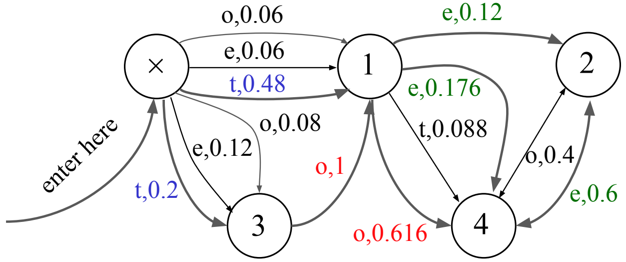
- Generate the next stage i+1 as before (except now <u>arcs</u> generate output, thus use only those arcs marked by the output symbol y_{i+1})

- For each generated *state*, compute $\alpha(state,i+1)$ $= \Sigma_{\text{inc. arcs}} P_Y(y_{i+1}|state, prev.state) \times \alpha(prev.state, i)$

position/stage: 0 1

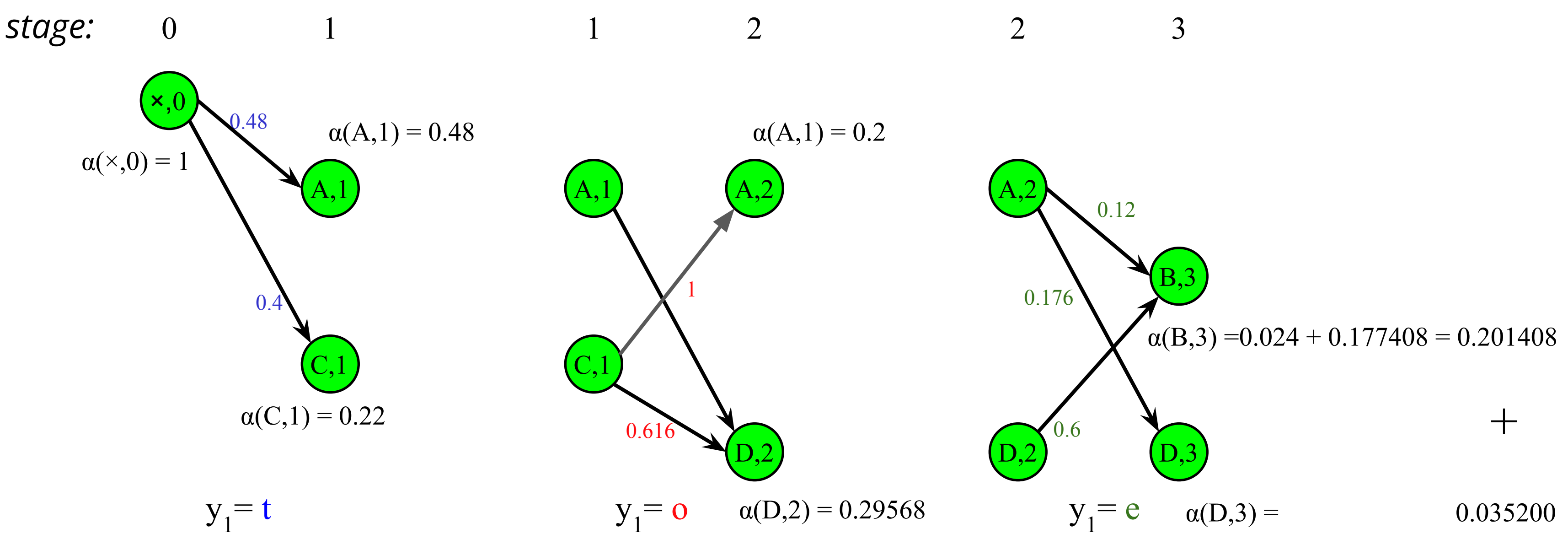


Y: t

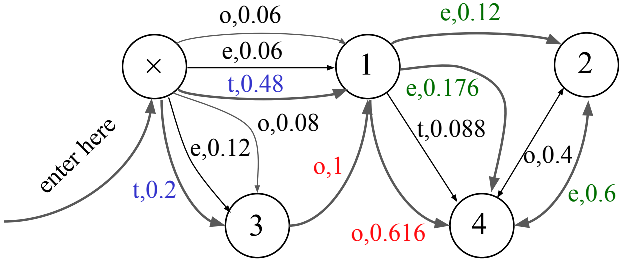


And forget about the previous state ...

Trellis: The Complete Example



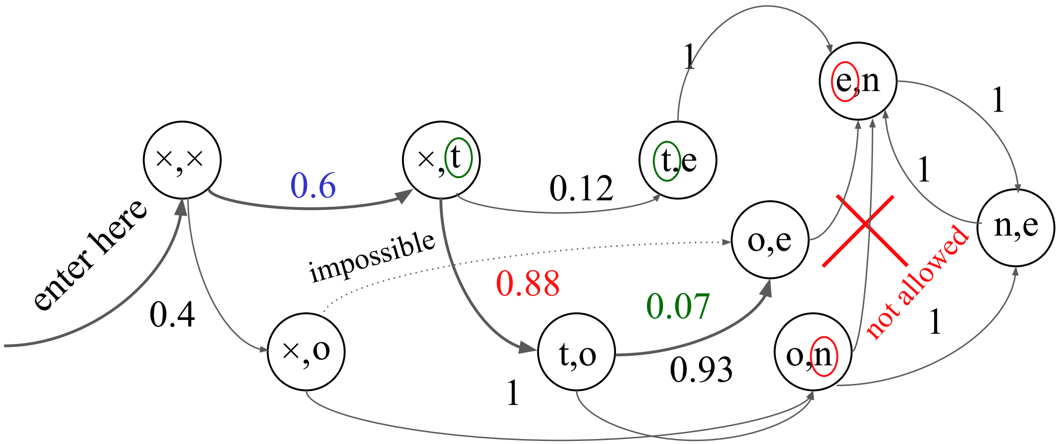
P(Y) = P(toe) = 0.236608

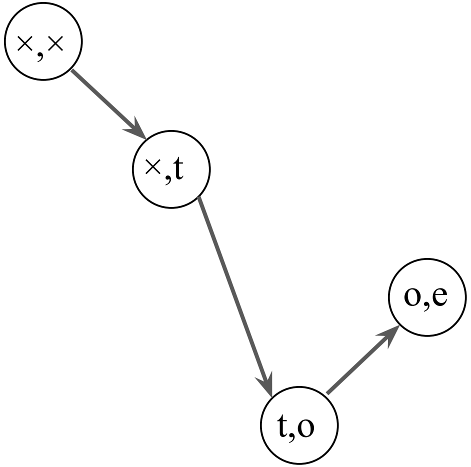


p(toe) = 0.48×0.616×0.6+
0.2×1×0.176 +
0.2×1×0.12 ≅ 0.237

The Case of Trigrams

- Like before, but:
  - states correspond to bigrams
  - output function always emits the second output symbol of the pair (state) to which the arc goes:





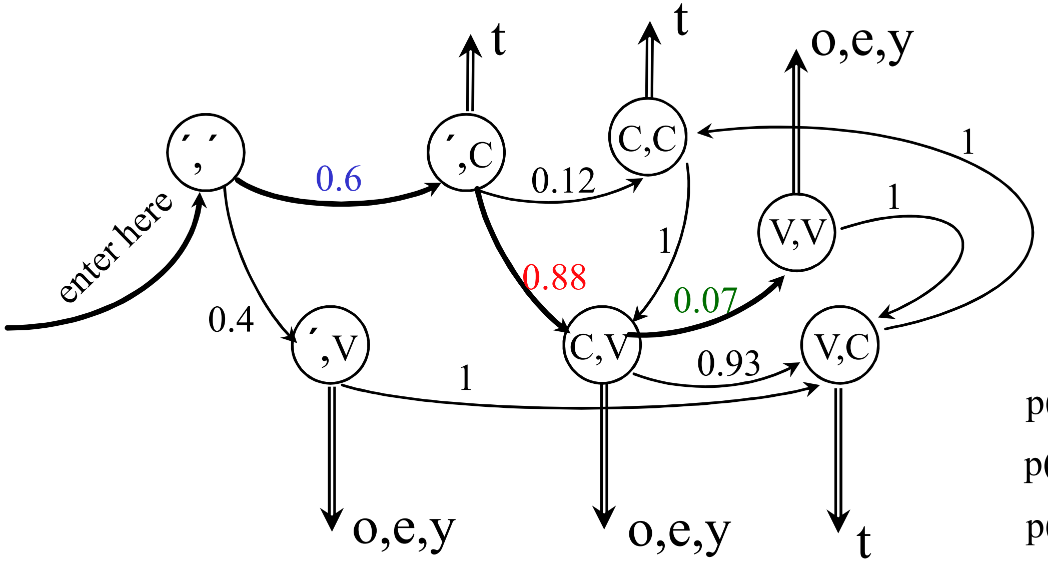
- Multiple paths not possible → trellis not really needed

Trigrams with Classes

- More interesting:
  - n-gram class LM: $p(w_i|w_{i-2},w_{i-1}) = p(w_i|c_i)\, p(c_i|c_{i-2},c_{i-1})$

    → states are pairs of classes (c_{i-1},c_i), and emit "words":

    (letters in our example)



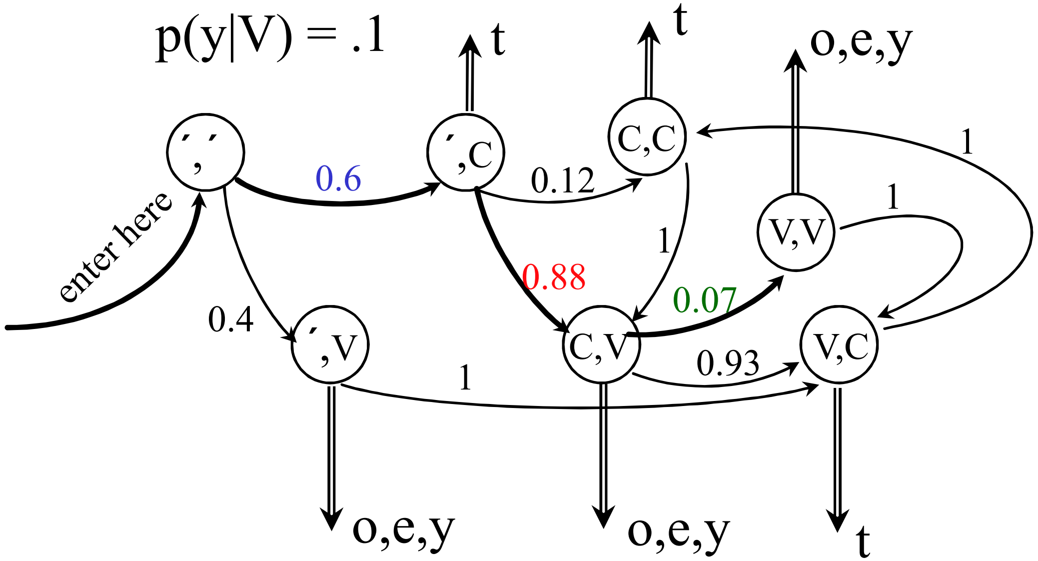
p(t|C) = 1 usual,
p(o|V) = .3 non-
p(e|V) = .6 overlapping
p(y|V) = .1 classes

p(toe) = .6 ´ 1 ´ .88 ´ .3 ´ .07 ´ .6 ≅ .00665
p(teo) = .6 ´ 1 ´ .88 ´ .6 ´ .07 ´ .3 ≅ .00665
p(toy) = .6 ´ 1 ´ .88 ´ .3 ´ .07 ´ .1 ≅ .00111
p(tty) = .6 ´ 1 ´ .12 ´ 1 ´ 1 ´ .1 ≅ .0072

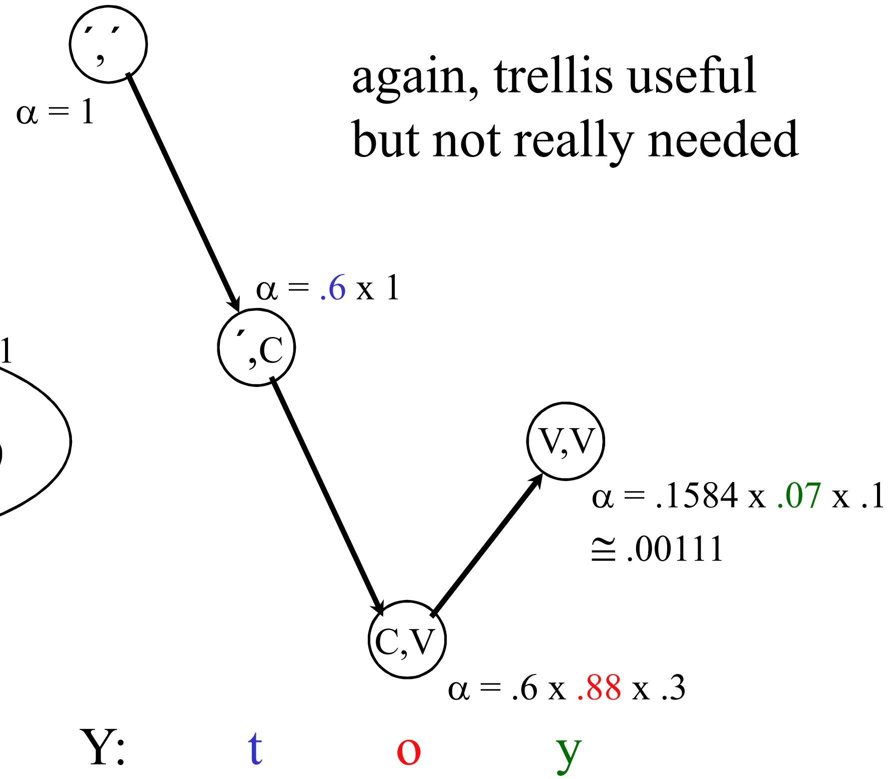
Class Trigrams: the Trellis

- Trellis generation (Y = “toy”):

$p(t|C) = 1$
$p(o|V) = .3$
$p(e|V) = .6$
$p(y|V) = .1$

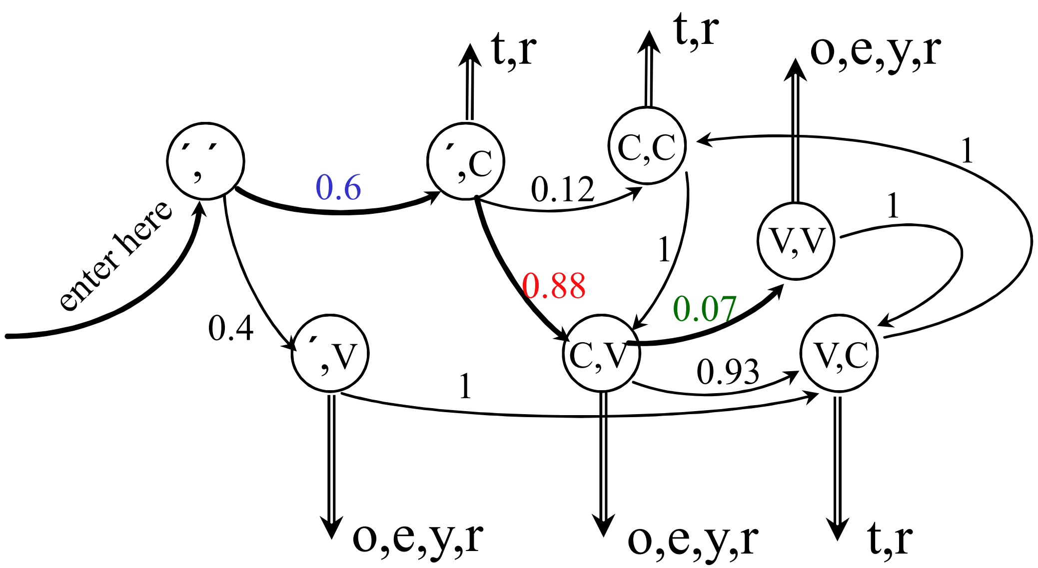


again, trellis useful
but not really needed



Overlapping Classes

- Imagine that classes may overlap
  - e.g. 'r' is sometimes vowel sometimes consonant, belongs to V as well as C:

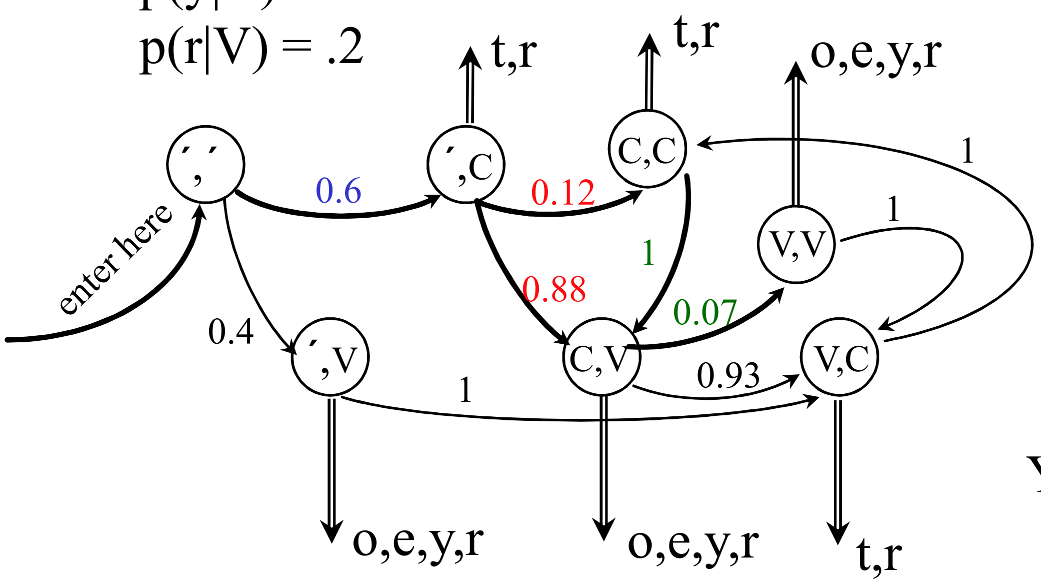


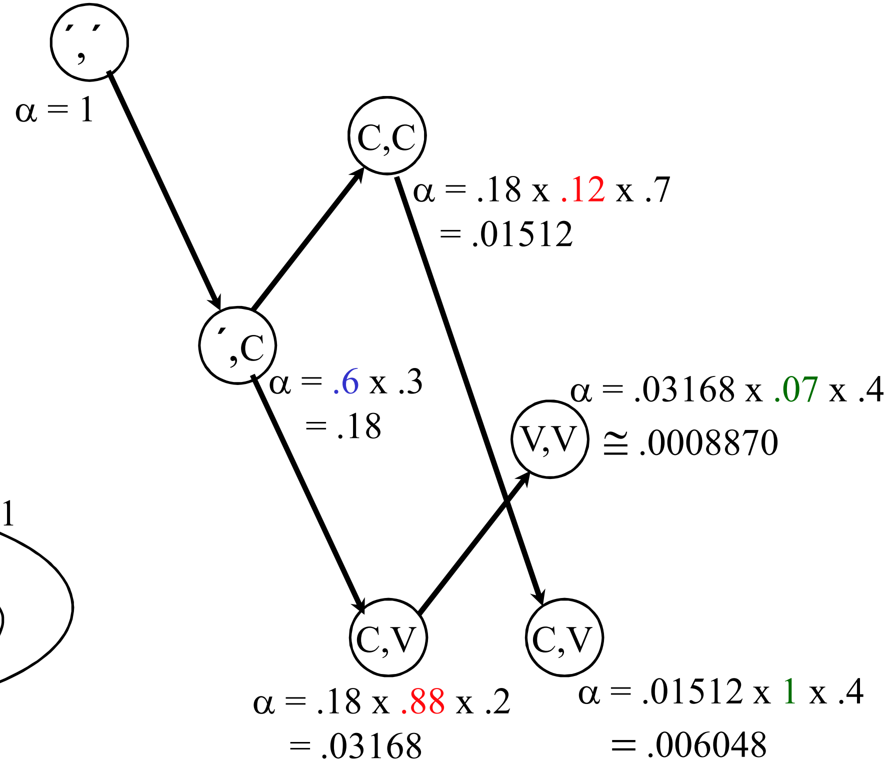
$p(t|C) = .3$
$p(r|C) = .7$
$p(o|V) = .1$
$p(e|V) = .3$
$p(y|V) = .4$
$p(r|V) = .2$

$p(try) = ?$

Overlapping Classes: Trellis Example

$p(t|C) = .3$
$p(r|C) = .7$
$p(o|V) = .1$
$p(e|V) = .3$
$p(y|V) = .4$
$p(r|V) = .2$





Y: t r y $p(Y) = \underline{.006935}$

Trellis: Remarks

- So far, we went left to right (computing α)
- Same result: going right to left (computing β)
  - supposed we know where to start (finite data)
- In fact, we might start in the middle going left <u>and</u> right
- Important for parameter estimation

  (Forward-Backward Algortihm alias Baum-Welch)
- Implementation issues:
  - scaling/normalizing probabilities, to avoid too small numbers & addition problems with many transitions

The Viterbi Algorithm

- Solving the task of finding the most likely sequence of states which generated the observed data
- i.e., finding

$$S_{best} = \text{argmax}_S P(S|Y)$$

which is equal to (Y is constant and thus P(Y) is fixed):

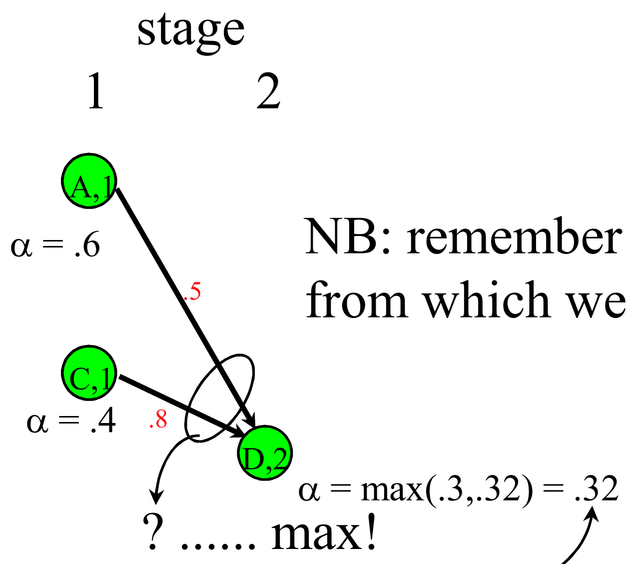
$$S_{best} = \text{argmax}_S P(S,Y) =$$

$$= \text{argmax}_S P(s_0,s_1,s_2,\ldots,s_k,y_1,y_2,\ldots,y_k) =$$

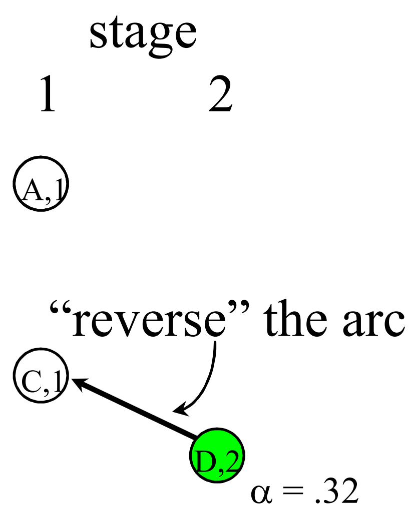
$$= \text{argmax}_S \Pi_{i=1..k}\, p(y_i|s_i,s_{i-1})p(s_i|s_{i-1})$$

The Crucial Observation

- Imagine the trellis build as before (but do not compute the αs yet; assume they are o.k.); stage *i*:



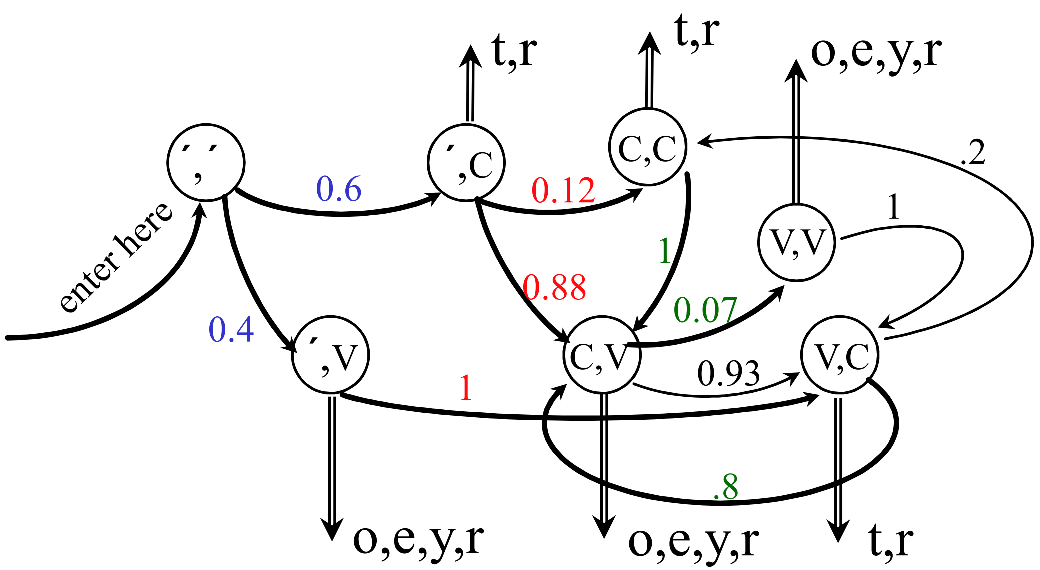
NB: remember previous state from which we got the maximum:



this is certainly the "backwards" maximum to (D,2)... but
it <u>cannot change even whenever we go forward</u> (M. Property: Limited History)

Viterbi Example

- 'r' classification (C or V?, sequence?):



$p(t|C) = .3$
$p(r|C) = .7$
$p(o|V) = .1$
$p(e|V) = .3$
$p(y|V) = .4$
$p(r|V) = .2$

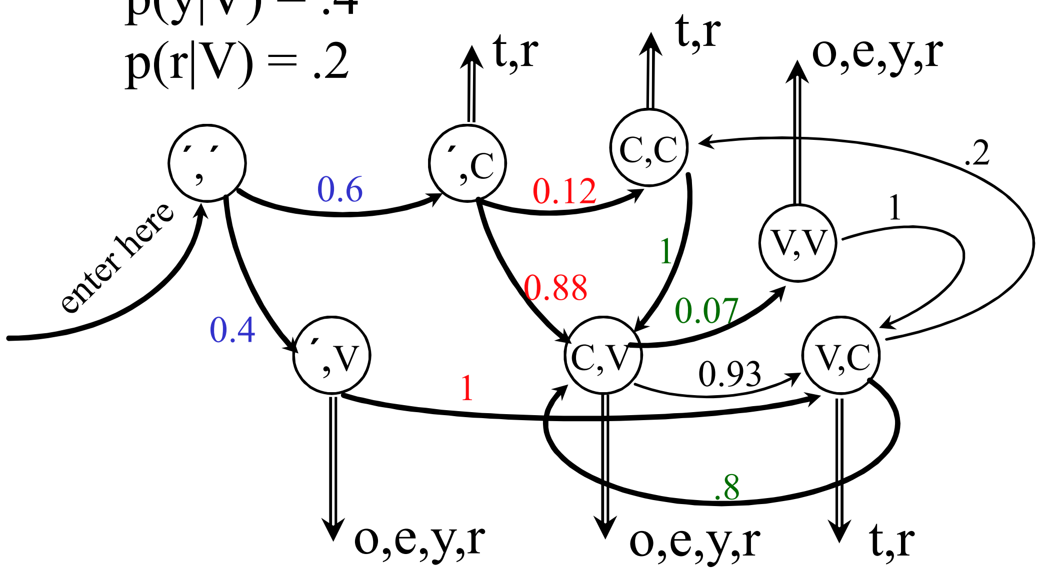
$\text{argmax}_{XYZ}\ p(rry|XYZ) = ?$

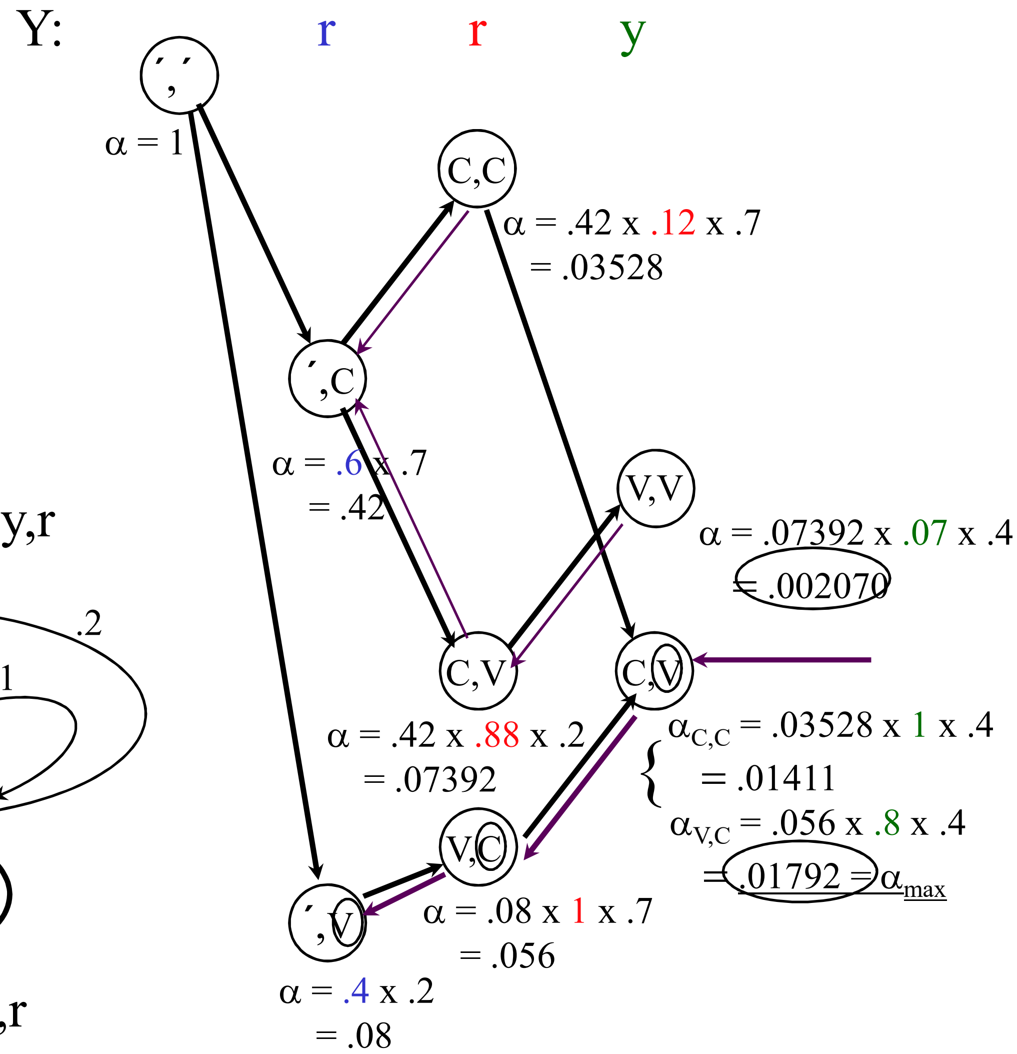
Possible state seq.: (´,v)(v,c)(c,v)[VCV], (´,c)(c,c)(c,v)[CCV], (´,c)(c,v)(v,v) [CVV]

Viterbi Computation

α in trellis state: best prob from start to here

$p(t|C) = .3$
$p(r|C) = .7$
$p(o|V) = .1$
$p(e|V) = .3$
$p(y|V) = .4$
$p(r|V) = .2$





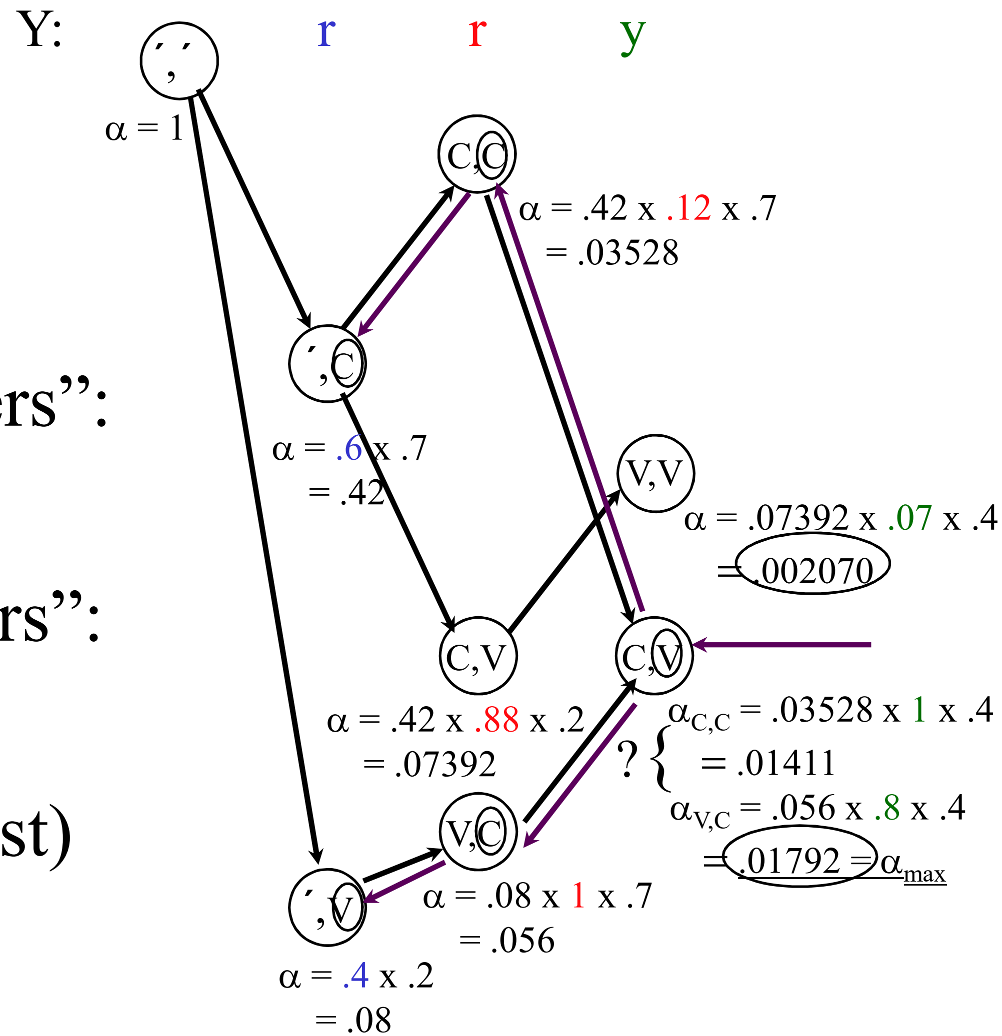
n-best State Sequences

- Keep track of n best "back pointers":
- Ex.: n= 2:

  Two "winners":

  VCV (best)

  CCV (2^{nd} best)

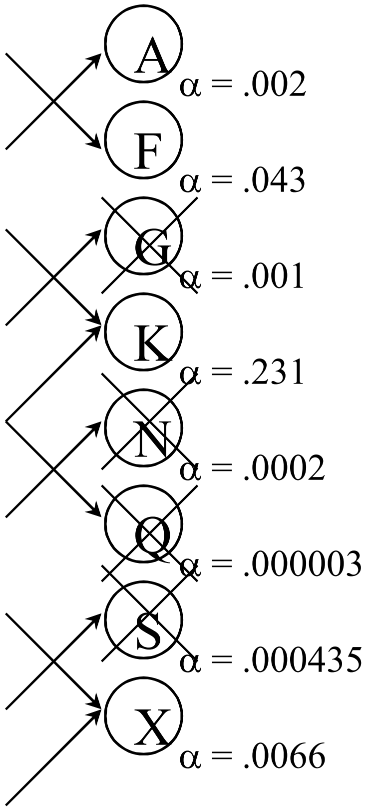


Tracking Back the n-best paths

- Backtracking-style algorithm:
  - **Start at the end, in the best of the n states (s_{best})**
  - **Put the other n-1 best nodes/back pointer pairs on stack, except those leading from s_{best} to the same best-back state.**
- Follow the back "beam" towards the start of the data, spitting out nodes on the way (backwards of course) using always only the <u>best</u> back pointer.
- At every beam split, push the diverging node/back pointer pairs onto the stack (node/beam width is sufficient!).
- When you reach the start of data, close the path, and pop the top-most node/back pointer(width) pair from the stack.
- Repeat until the stack is empty; expand the result tree if necessary.

Pruning

- Sometimes, too many trellis states in a stage:



criteria: (a) α < threshold
(b) $\Sigma\pi$ < threshold
(c) # of states > threshold
(get rid of smallest α)

HMM Parameter Estimation: the Baum-Welch Algorithm

HMM: The Tasks

- HMM (the general case):
  - five-tuple (S, S_0, Y, P_S, P_Y), where:
    - **$S = \{s_1,s_2,\ldots,s_T\}$ is the set of states, S_0 is the initial state,**
    - **$Y = \{y_1,y_2,\ldots,y_V\}$ is the output alphabet,**
    - **$P_S(s_j|s_i)$ is the set of prob. distributions of transitions,**
    - **$P_Y(y_k|s_i,s_j)$ is the set of output (emission) probability distributions.**
- Given an HMM & an output sequence $Y = \{y_1,y_2,\ldots,y_k\}$:

  ✓(Task 1) compute the probability of Y;

  ✓(Task 2) compute the most likely sequence of states which has generated Y.

  (Task 3) Estimating the parameters (transition/output distributions)

A Variant of EM

- Idea (~ EM, for another variant see LM smoothing):
  - Start with (possibly random) estimates of P_S and P_Y.
  - Compute (fractional) "counts" of state transitions/emissions taken, from P_S and P_Y, given data Y.
  - Adjust the estimates of P_S and P_Y from these "counts" (using the MLE, i.e. relative frequency as the estimate).
- Remarks:
  - many more parameters than the simple four-way smoothing
  - no proofs here; see Jelinek, Chapter 9

Setting

- HMM (without P_S, P_Y) (S, S_0, Y), and data $T = \{y^i \in Y\}_{i=1..|T|}$
    - **will use T ~ |T|**
  - HMM structure is given: (S, S_0)
  - P_S:Typically, one wants to allow "fully connected" graph
    - **(i.e. no transitions forbidden ~ no transitions set to hard 0)**
    - **why? → we better leave it on the learning phase, based on the data!**
    - **sometimes possible to remove some transitions ahead of time**
  - P_Y: should be restricted (if not, we will not get anywhere!)
    - **restricted ~ hard 0 probabilities of p(y|s,s')**
    - **"Dictionary": states ↔ words, "m:n" mapping on S × Y (in general)**

Initialization

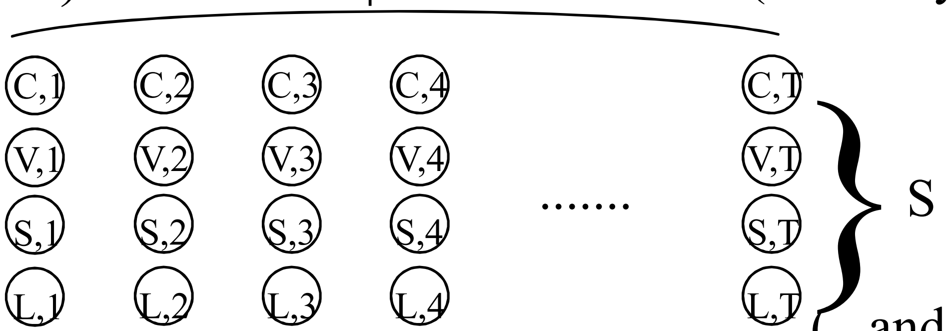
- For computing the initial expected "counts"
- Important part
  - EM guaranteed to find a *local* maximum only (albeit a good one in most cases)
- P_Y initialization more important
  - fortunately, often easy to determine
    - **together with dictionary ↔ vocabulary mapping, get counts, then MLE**
- P_S initialization less important
  - e.g. uniform distribution for each p(.|s)

Data Structures

- Will need storage for:
  - The predetermined structure of the HMM

    (unless fully connected → need not to keep it!)
  - The parameters to be estimated (P_S, P_Y)
  - The expected counts (same size as P_S, P_Y)
  - The training data $T = \{y^i \in Y\}_{i=1..T}$
  - The trellis (if f.c.): ↑ T Size: T ´ S (Precisely, |T|´|S|)

Each trellis state:
two [float] numbers
(forward/backward)



(...and then some)

The Algorithm Part I

1. Initialize P_S, P_Y
2. Compute "forward" probabilities:
   - **follow the procedure for trellis (summing), compute $\alpha(s,i)$**
   - **use the current values of P_S, P_Y ($p(s'|s)$, $p(y|s,s')$):**

     $$\alpha(s',i) = \sum_{s \rightarrow s'} \alpha(s,i-1) \times p(s'|s) \times p(y_i|s,s')$$

   - **NB: do not throw away the previous stage!**
3. Compute "backward" probabilities
   - **start at all nodes of the last stage, proceed backwards, $\beta(s,i)$**
   - **i.e., probability of the "tail" of data from stage *i* to the end of data**

     $$\beta(s',i) = \sum_{s \leftarrow s'} \beta(s,i+1) \times p(s|s') \times p(y_{i+1}|s',s)$$

   - **also, keep the $\beta(s,i)$ at all trellis states**

The Algorithm Part II

4. Collect counts:
   – for each output/transition pair compute

$$c(y,s,s') = \Sigma_{i=0..k-1,y=y_{i+1}} \alpha(s,i)\ \underbrace{p(s'|s)\ p(y_{i+1}|s,s')}_{\text{this transition prob ´ output prob}}\ \beta(s',i+1)$$

one pass through data, only stop at (output) y

prefix prob.

tail prob

$$c(s,s') = \Sigma_{y \in Y}\ c(y,s,s') \text{ (assuming all observed } y_i \text{ in Y)}$$

$$c(s) = \Sigma_{s' \in S}\ c(s,s')$$

5. Reestimate: $p'(s'|s) = c(s,s')/c(s)$ $\quad p'(y|s,s') = c(y,s,s')/c(s,s')$

6. Repeat 2-5 until desired convergence limit is reached.

Baum-Welch: Tips & Tricks

- Normalization badly needed
  - long training data $\rightarrow$ extremely small probabilities
- Normalize α,β using the same norm. factor:

$$N(i) = \Sigma_{s \in S}\ \alpha(s,i)$$

as follows:

- **compute $\alpha(s,i)$ as usual (Step 2 of the algorithm), computing the sum N(i) at the given stage *i* as you go.**
- **at the end of each stage, recompute all αs (for each state s):**

□ $$\alpha^*(s,i) = \alpha(s,i) / N(i)$$

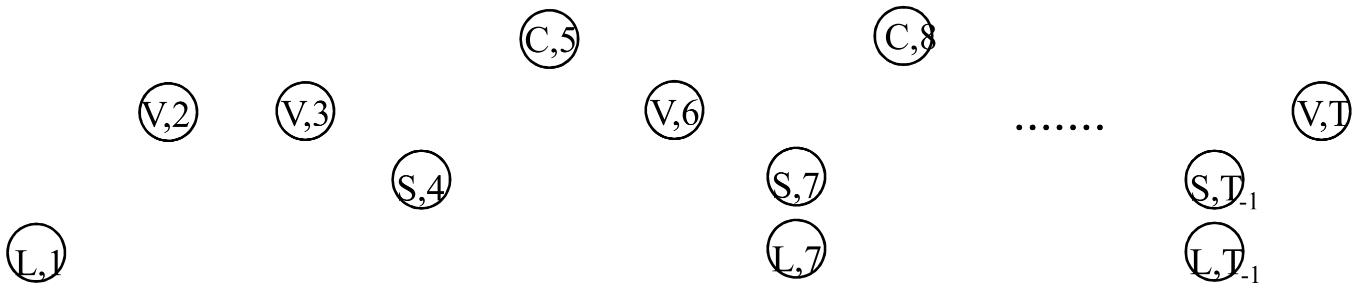
- **use the same N(i) for βs at the end of each backward (Step 3) stage:**

□ $$\beta^*(s,i) = \beta(s,i) / N(i)$$

Example

- Task: pronunciation of “the”
- Solution: build HMM, fully connected, 4 states:
  - **S - short article, L - long article, C,V - starting w/consonant, vowel**
  - **thus, only “the” is ambiguous (a, an, the - not members of C,V)**
- Output from states only ($p(w|s,s') = p(w|s')$)
- Data Y: an egg and a piece of the big .... the end

Trellis:



Example: Initialization

- Output probabilities:

  $p_{init}(w|c) = c(c,w) / c(c)$; where $c(S,the) = c(L,the) = c(the)/2$

  (other than that, everything is deterministic)
- Transition probabilities:
  - $p_{init}(c'|c) = 1/4$ (uniform)
- Don't forget:
  - about the space needed
  - initialize $\alpha(X,0) = 1$ (X : the never-occurring front buffer st.)
  - initialize $\beta(s,T) = 1$ for all s (except for s = X)

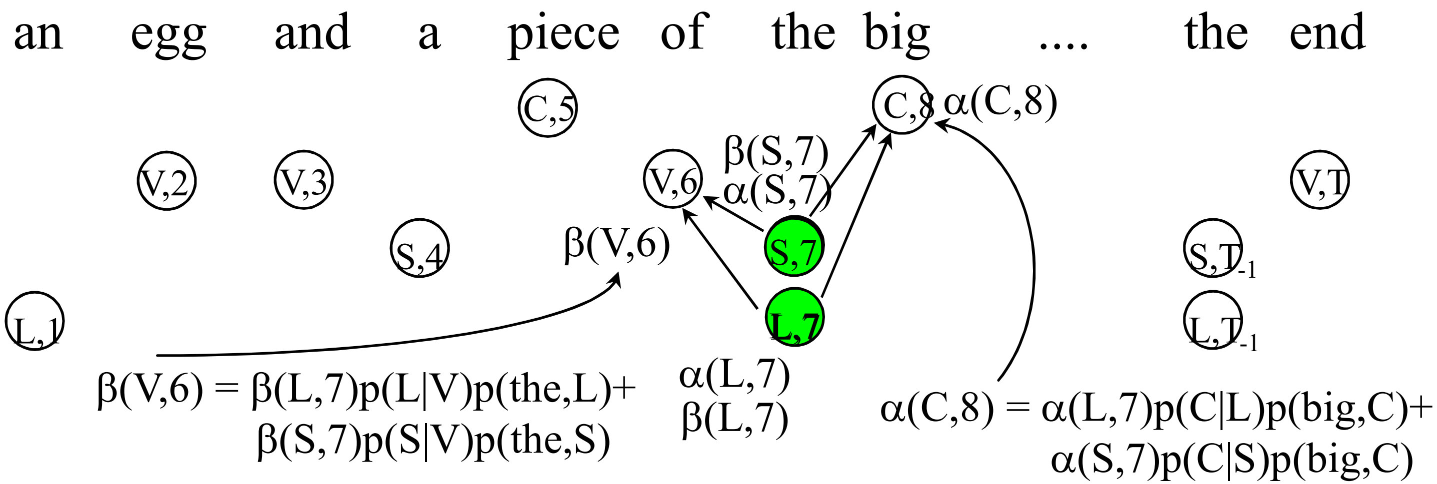
Fill in alpha, beta

- Left to right, alpha:

$$\boldsymbol{\alpha(s',i) = \Sigma_{s \to s'} \alpha(s,i\text{-}1) \times p(s'|s) \times p(w_i|s')}$$

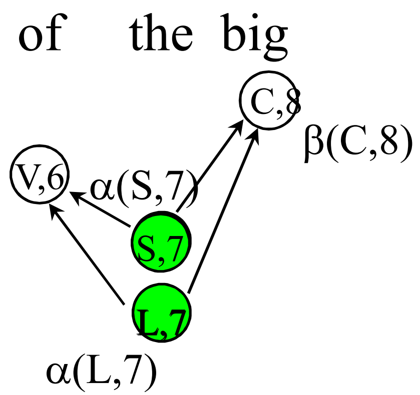
output from states

- Remember normalization (N(i)).
- Similarly, beta (on the way back from the end).



Counts & Reestimation

- One pass through data
- At each position *i*, go through all pairs (s_i, s_{i+1})
- Increment appropriate counters by frac. counts (Step 4):
  - **$inc(y_{i+1}, s_i, s_{i+1}) = a(s_i, i)\ p(s_{i+1}|s_i)\ p(y_{i+1}|s_{i+1})\ b(s_{i+1}, i+1)$**
  - **$c(y, s_i, s_{i+1})$ += inc (for y at pos i+1)**
  - **$c(s_i, s_{i+1})$ += inc (always)**
  - **$c(s_i)$ += inc (always)**



inc(big,L,C) = $\alpha(L,7)p(C|L)p(big,C)\beta(C,8)$

inc(big,S,C) = $\alpha(S,7)p(C|S)p(big,C)\beta(C,8)$

- Reestimate p(s'|s), p(y|s)
  - **and hope for increase in p(C|S) and p(V|L)...!!**

HMM: Final Remarks

- Parameter “tying”:
  - keep certain parameters same (~ just one “counter” for all of them)
  - any combination in principle possible
  - ex.: smoothing (just one set of lambdas)
- Real Numbers Output
  - Y of infinite size (R, R^n):
    - **parametric (typically: few) distribution needed (e.g., “Gaussian”)**
- “Empty” transitions: do not generate output
    - **~ vertical arcs in trellis; do not use in “counting”**