Statistical Methods in Natural Language Processing

6. Mutual Information and Word Classes

Pavel Pecina, Jindřich Helcl

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Charles University
Faculty of Mathematics and Physics
Institute of Formal and Applied Linguistics

Course Segments

1. Introduction, probability, essential information theory
2. Statistical language modelling (n-gram)
3. Statistical properties of words
4. Word representations
5. Hidden Markov models, Tagging

Recap from Last Week

Collocations

- J. R. Firth (1957):
  - *"You shall know a word by the company it keeps."*
  - *"Collocations of a given word are statements of the habitual or customary places of that word."*
- M&S (Chapter 5):
  - *"A collocation is an expression consisting of two or more words that correspond to some conventional way of saying things."*
- Examples:
  - *strong tea, weapons of mass destruction*
  - *to make up, the rich and powerful*
- Valid or invalid?
  - *a stiff breeze,* but not *a stiff wind*
  - *broad daylight,* but not *bright daylight*

Properties of Collocations

- Typical properties/criterions of collocations:
  - non-compositionality
  - non-substitutability
  - non-modifiability
- Collocations usually cannot be translated word-by-word
  - e.g. *take a shower* → *osprchovat se,* but not *vzít sprchu*
- A phrase can be a collocation even if it is not consecutive
  - e.g. *knock … door*

Content from Rada Mihalcea

How to Find Collocations?

- Frequency (simplest method)
  - plain
  - filtered
- Mean and variance of the distance between focal word and collocating word
- Hypothesis testing
  - t test
  - $\chi 2$ test
- Pointwise Mutual Information

Hypothesis Testing

- Two words can co-occur by chance
  - High frequency and low variance can be accidental
- Hypothesis Testing measures the confidence that this co-occurrence was really due to association, and not just due to chance.
- Formulate a null hypothesis H_0 that there is no association between the words beyond chance occurrences:
  - H_0 states what should be true if two words do not form a collocation.
  - If H_0 can be rejected, the words do not co-occur by chance, and they form a collocation
- Compute the probability p that the event would occur if H_0 were true:
  - reject H_0 if p is too low (typically beneath a significance level of $p < 0.05, 0.01, \ldots$)
  - retain H_0 as possible otherwise.

t-test (Student's *t*-test)

- The test looks at the difference between the observed and expected means, scaled by the variance of the data, and tells us how likely one is to get a sample of that mean and variance, assuming that the sample is drawn from a normal distribution with mean μ.

$$t = \frac{\bar{x} - \mu}{\sqrt{\frac{s^2}{N}}}$$

  - Where $\bar{x}$ is the real data mean (observed in data)
  - s^2 is the variance
  - N is the sample size
  - μ is the mean of the distribution (expected under H_0)

χ^2 test: An example

- Observed occurrences:

	$w_1 = new$	$w_1 \neq new$
$w_2 = companies$	8 (*new companies*)	4667 (e.g., *old companies*)
$w_2 \neq companies$	15820 (e.g., *new machines*)	14287181 (e.g., *old machines*)

- The χ^2 statistic sums differences between observed and expected values in all cells of the table, scaled by the magnitude of the expected values:

$$X^2 = \sum_{i,j} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

  - where i ranges over rows of the table, j ranges over columns, O_{ij} is the observed value for cell (i, j) and E_{ij} is the expected value.

χ^2 test: A Simpler Form

- χ^2 test can be applied to tables of any size, but it has a simpler form for 2-by-2 tables (i.e. bigram collocations):

$$\chi^2 = \frac{N(O_{11}O_{22} - O_{12}O_{21})^2}{(O_{11} + O_{12})(O_{11} + O_{21})(O_{12} + O_{22})(O_{21} + O_{22})}$$

Pointwise Mutual Information

- An information-theoretically motivated measure for discovering interesting collocations is pointwise mutual information
- It is a measure of how much one word tells us about the other:

$$\mathrm{PMI}(x, y) = \log \frac{P(x, y)}{P(x)\,P(y)}$$

“observed”

“expected (under H_0)”

  - $P(x,y)$ is the joint probability of x and y occurring together,
  - $P(x)$ and $P(y)$ are the individual probabilities of x and y

- PMI is ***<u>NOT</u>*** the MI as defined in Information Theory (MI is the average of PMI)

Mutual Information and Word Classes

The Problem

Not enough data:

- Language Modeling: we do not see “correct” n-grams
  - solution so far: smoothing
- suppose we see:
  - *short homework, short assignment, simple homework*
- but not:
  - *simple assignment*
- What happens to our (bigram) LM?
  - $p(\textit{homework}|\textit{simple})$ = high probability
  - $p(\textit{assignment}|\textit{simple})$ = low probability (smoothed with $p(\textit{assignment})$)
- They should be much closer!

Word Classes

Observation:

- similar words behave in a similar way
- i.e. appear in similar context (the Distributional Hypothesis)

- Assuming trigram language model:
  - *a … homework* (any attribute of homework: *short, simple, late, difficult*),
  - *… the woods* (any verb that has the woods as an object: *walk, cut, save)*
  - *a (short,long,difficult,…) (homework,assignment,task,job,…)*

Solution

Use the Word Classes as the “reliability” measure:

- Example: we see
  - *short homework, short assignment, simple homework*
- but not:
  - *simple assigment*

- Cluster into classes:
  - *(short, simple) (homework, assignment)*
  - covers “*simple assignment*”, too
- Gaining:
  - realistic estimates for unseen n-grams
- Losing:
  - accuracy (level of detail) within classes

The New Model

Rewrite the n-gram LM using classes:

- Original definition: $[k = 1 \ldots n]$

  $p_k(w_i|h_i) = c(h_i,w_i) / c(h_i)$ [history: $(k\text{-}1)$ words]

- Introduce classes:

$$\boxed{p_k(w_i|h_i) = p(w_i|c_i)\, p_k(c_i|h_i)}$$

  - history: classes, too: [for trigram: $h_i = c_{i-2},c_{i-1}$, bigram: $h_i = c_{i-1}$]
- Smoothing as usual
  - over $p_k(w_i|h_i)$, where each is defined as above (except uniform which is $1/|V|$)

Training Data

- Suppose we already have a mapping:
  - $r: V \rightarrow C$ assigning each word its class ($c_i = r(w_i)$)
- Expand the training data:
  - $T = (w_1, w_2, \ldots, w_{|T|})$ into
  - $T_C = (\langle w_1,r(w_1)\rangle, \langle w_2,r(w_2)\rangle, \ldots, \langle w_{|T|},r(w_{|T|})\rangle)$
- Effectively, we have two streams of data:
  - word stream: $w_1, w_2, \ldots, w_{|T|}$
  - class stream: $c_1, c_2, \ldots, c_{|T|}$ (def. as $c_i = r(w_i)$)
- Expand Heldout, Test data too

Training the New Model

- Using ML estimates (as expected):
  - $p(w_i|c_i) = p(w_i|r(w_i)) = c(w_i) / c(r(w_i)) = c(w_i) / c(c_i)$
    - !!! $c(w_i,c_i) = c(w_i)$ [since c_i determined by w_i]
  - $p_k(c_i|h_i)$:
    - $p_3(c_i|h_i) = p_3(c_i|c_{i-2}, c_{i-1}) = c(c_{i-2}, c_{i-1}, c_i) / c(c_{i-2}, c_{i-1})$
    - $p_2(c_i|h_i) = p_2(c_i|c_{i-1}) = c(c_{i-1}, c_i) / c(c_{i-1})$
    - $p_1(c_i|h_i) = p_1(c_i) = c(c_i) / |T|$
- Then smooth as usual
  - not the $p(w_i|c_i)$ nor $p_k(c_i|h_i)$ individually, but the $p_k(w_i|h_i)$

Classes: How To Get Them

- We supposed the classes are given
- Maybe there are in (human) dictionaries, but…
  - dictionaries are incomplete
  - dictionaries are unreliable
  - do not define classes as equivalence relation (overlap)
  - do not define classes suitable for LM
    - small, short… maybe; small and difficult?

→ we have to construct them <u>from data</u> (again…)

Creating the Word-to-Class Map (Brown's Classes)

- Consider <u>bigram</u> model for now.
- Bigram estimate:

  $$p_2(c_i|h_i) = p_2(c_i|c_{i-1}) = c(c_{i-1},c_i) / c(c_{i-1}) = c(r(w_{i-1}),r(w_i)) / c(r(w_{i-1}))$$

- Form of the model:
  - just raw bigram for now:

    $$P(T) = \Pi_{i=1..|T|} p(w_i|r(w_i))\ p_2(r(w_i)|r(w_{i-1})) \qquad (p_2(c_1|c_0) =_{df} p(c_1))$$

- Maximize over r (given $r \rightarrow$ fixed p, p_2):
  - define objective

    $$L(r) = 1/|T|\ \Sigma_{i=1..|T|} \log(p(w_i|r(w_i))\ p_2(r(w_i))|r(w_{i-1})))$$

    $r_{best} = \text{argmax}_r\ L(r)$ ($L(r)$ = norm. logprob of training data ... as usual)

Simplifying the Objective Function

- Start from $L(r) = 1/|T| \Sigma_{i=1..|T|}\log(p(w_i|r(w_i))\ p_2(r(w_i)|r(w_{i-1})))$:

  $1/|T| \Sigma_{i=1..|T|}\log(p(w_i|r(w_i))\ \underline{p(r(w_i))}\ p_2(r(w_i)|r(w_{i-1}))\ /\ \underline{p(r(w_i)))} =$

  $1/|T| \Sigma_{i=1..|T|}\log(\underline{p(w_i,r(w_i)})\ )\ p_2(r(w_i)|r(w_{i-1}))\ /\ p(r(w_i))) =$

  $1/|T| \Sigma_{i=1..|T|}\log(\underline{p(w_i)}) + 1/|T| \Sigma_{i=1..|T|}\log(p_2(r(w_i)|r(w_{i-1}))\ /\ p(r(w_i))) =$

  $-H(W) + 1/|T| \Sigma_{i=1..|T|}\log(p_2(r(w_i)|r(w_{i-1}))\ \underline{p(r(w_{i-1}))\ /\ (p(r(w_{i-1}))}\ p(r(w_i))))$

  $=$

  $-H(W) + 1/|T| \Sigma_{i=1..|T|}\log(\underline{p(r(w_i),r(w_{i-1})})\ )\ /\ (p(r(w_{i-1}))\ p(r(w_i)))) =$

  $-H(W) + \Sigma_{d,e\in C}\ p(d,e)\ \log(\ p(d,e)\ /\ (p(d)\ p(e))\ ) =$

  $-H(W) + I(D,E)$ (event E picks class adjacent (to the right) to the one picked by D)

- Since W does not depend on r, we ended up with I(D,E).

Maximizing Mutual Information (dependent on mapping r)

- Result from previous slide:
  - Maximizing the probability of data amounts to maximizing $I(D,E)$, the Mutual Information of the adjacent classes.
- Good:
  - We know what a MI is, and we know how to maximize.
- Bad:
  - There is no way how to maximize over so many possible partitionings: $|V|^{|V|}$ - no way to test them all.

Training or Heldout?

- Training:
  - best $I(D,E)$: all words in a class of its own
    $\rightarrow$ <u>will not give us anything new.</u>
- Heldout: ok, but:
  - must smooth to test <u>any</u> possible partitioning (unfeasible):
    $\rightarrow$ <u>using raw model: 0 probability of heldout (almost) guaranteed</u>
    $\rightarrow$ will not be able to compare anything
- Solution:
  - use training anyway, but only keep $I(D,E)$ as large as possible

The Greedy Algorithm

- Define merging operation on the mapping $r: V \rightarrow C$:
  - merge: $R \times C \times C \rightarrow R' \times C\text{-}1$: $(r,k,l) \rightarrow r',C'$ such that
  - $C^{-1} = \{C - \{k,l\} \cup \{m\}\}$ (throw out k and l, add new $m \notin C$)
  - $r'(w) = m$ for $w \in r_{INV}(\{k,l\})$,
    $r(w)$ otherwise.

1. Start with each word in its own class ($C = V$), $r = id$.
2. Merge two classes k,l into one, m, such that
   $(k,l) = \text{argmax}_{k,l}\ I_{merge(r,k,l)}(D,E)$.
3. Set new $(r,C) = merge(r,k,l)$.
4. Repeat 2 and 3 until $|C|$ reaches predetermined size.

Brown’s Classes: Programming Tips & Tricks

Complexity Issues

Still too complex:

- $|V|$ iterations of the steps 2 and 3.
- $|V|^2$ steps to maximize $\mathrm{argmax}_{k,l}$ (selecting k,l freely from $|C|$, which is in the order of $|V|^2$)
- $|V|^2$ steps to compute $I(D,E)$ (sum within sum, all classes, also: includes $\log$)

$\Rightarrow$ total: $|V|^5$

- i.e., for $|V| = 100$, about 10^{10} steps (several hours!)
- but $|V| \sim 50{,}000$ or more

Formula breakdown

- Mutual Information at k^{th} iteration (= k classes):
  - $I_k = \sum_{l,r \in C} p_k(l,r) \log(p_k(l,r) / (p_{kl}(l)\, p_{kr}(r)))$
- For each pair of classes at iteration k, we define:
  - $q_k(l,r) = p_k(l,r) \log(p_k(l,r) / (p_{kl}(l)\, p_{kr}(r)))$
- So:
  - $I_k = \sum_{l,r \in C} q_k(l,r)$
- $q_k(l,r)$ using bigram counts $c_k(l,r)$:

  $q_k(l,r) = c_k(l,r)/N \log(N\, c_k(l,r)/(c_{kl}(l)\, c_{kr}(r)))$

l \ r	c_1	c_2	c_3	c_4
c_1	10	2	0	1
c_2	0	0	5	2
c_3	0	2	0	3
c_4	2	3	0	0

unigram/marginal counts

Trick #1: Recomputing MI the Smart Way

- For test-merging c_2 and c_4 we recompute only rows/columns 2 & 4:

k:

$l \setminus r$	c_1	c_2	c_3	c_4
c_1	10	2	0	1
c_2	0	0	5	2
c_3	0	2	0	3
c_4	2	3	0	0

Trick #1: Recomputing MI the Smart Way

- For test-merging c_2 and c_4 we recompute only rows/columns 2 & 4:
    1. Subtract column/row (2 & 4) from the MI sum:

k:

$l \backslash r$	c_1	c_2	c_3	c_4
c_1	10	2	0	1
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    (be careful at the intersections)

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- For test-merging c_2 and c_4 we recompute only rows/columns 2 & 4:

1. Subtract column/row (2 & 4) from the MI sum:

   (be careful at the intersections)

k:

1 \ r	c_1	c_2	c_3	c_4
c_1	10	2	0	1
c_2	0	0	5	2
c_3	0	2	0	3
c_4	2	3	0	0

2. Add sums of the merged counts (row & column for (c_2' is the merged class):

   (watch the intersection again)

k-1:

1 \ r	c_1	c_2'	c_3
c_1	10	3	0
c_2'	2	5	5
c_3	0	5	0

Trick #2: Precompute the Counts-to-be-Subtracted

- Summing loop goes through i,j
  - … but the single row/column sums do not depend on the (resulting sums after the) merge ⇒ <u>can be precomputed</u>
  - only $2k$ logs to compute at each algorithm iteration, instead of k^2
- Then for each “merge-to-be” compute only add-on sums, plus “intersection adjustment”

Formulas for Tricks #1 and #2

- Recap:

$$q_k(l,r) = p_k(l,r) \log(p_k(l,r) / (p_{kl}(l)\, p_{kr}(r)))$$

the same, but using counts:

$$q_k(l,r) = c_k(l,r)/N \log(N\, c_k(l,r)/(c_{kl}(l)\, c_{kr}(r)))$$

- Define further (row+column <u>a</u> sum):

$$s_k(a) = \Sigma_{l=1..k} q_k(l,a) + \Sigma_{r=1..k} q_k(a,r) - q_k(a,a)$$

Intersection adjustment

	c_2	c_3	c_4
c_2	0	5	2
c_3	2	0	3
c_4	3	0	0

- Then, the subtraction part of Trick #1 amounts to

$$sub_k(a,b) = s_k(a) + s_k(b) - q_k(a,b) - q_k(b,a)$$

precomputed

remaining intersection adjustment

Formulas - cont.

- After-merge add-on:

  $add_k(a,b) = \Sigma_{l=1..k,l\neq a,b} q_k(l,a+b) + \Sigma_{r=1..k,r\neq a,b} q_k(a+b,r) + q_k(a+b,a+b)$

- <u>a+b</u> is the <u>new (merged) class</u>
  - Hint: use the definition of q_k as a “macro”, and then:

    $p_k(a+b,r) = p_k(a,r) + p_k(b,r)$ (same for other sums, equivalent)
- The above sums cannot be precomputed
- Mutual Information after merge of class a,b:
  - $I_{k-1}(a,b) = I_k - sub_k(a,b) + add_k(a,b)$
  - I_k is the “old” MI, kept from previous iteration of the algorithm

Trick #3: Ignore Zero Counts

- Many bigrams are 0
  - e.g. in the Canadian Hansards corpus, < 0.1 % of bigrams are non-zero)
- Consider non-zero bigrams only:
  - e.g. create linked lists of non-zero counts in columns and rows
  - similar effect: use hashes (store non-zero-count bigrams)
- Update links after merge (after step 3)

Trick #4: Use Updated Loss of MI

- We are now down to $|V|^4$: $|V|$ merges, each merge takes $|V|^2$ "test-merges", each test-merge involves order-of-$|V|$ operations ($\mathrm{add}_k(i,j)$ term, slide 34)
- <u>Observation:</u>
  - many numbers (s_k, q_k) needed to compute the mutual information loss due to a merge of $i+j$ ***do not change*:** namely, those which are not in the vicinity of neither i nor j.
- <u>Idea:</u>
  - keep the **MI loss matrix** for all pairs of classes, and (after a merge) update only those cells which have been influenced by the merge.

Formulas for Trick #4 (s_{k-1}, L_{k-1})

- Keep a matrix of "losses" $L_k(d,e)$ [symmetry: $L_k(d,e) = L_k(e,d)$]
- Init: $L_k(d,e) = sub_k(d,e) - add_k(d,e)$ [then $I_{k-1}(d,e) = I_k - L_k(d,e)$]
- Suppose a,b are now the two classes merged into a
- Update (k-1: index used for the next iteration; $i,j \neq a,b$):

$$s_{k-1}(i) = s_k(i) - q_k(i,a) - q_k(a,i) - q_k(i,b) - q_k(b,i) + q_{k-1}(a,i) + q_{k-1}(i,a)$$

$$\begin{aligned} L_{k-1}(i,j) = {} & L_k(i,j) - s_k(i) + s_{k-1}(i) - s_k(j) + s_{k-1}(j) + \\ & + q_k(i+j,a) + q_k(a,i+j) + q_k(i+j,b) + q_k(b,i+j) - \\ & - q_{k-1}(i+j,a) - q_{k-1}(a,i+j) \end{aligned}$$

Completing Trick #4

- $s_{k-1}(a)$ must be computed using the "Init" sum (see the prev. slide).
- $L_{k-1}(a,i) = L_{k-1}(i,a)$ must be computed in a similar way, for all $i \neq a,b$.
- $s_{k-1}(b)$, $L_{k-1}(b,i)$, $L_{k-1}(i,b)$ are not needed anymore (keep track of such data, i.e. mark every class already merged into some other class and do not use it anymore).
- Keep track of the minimal loss during the $L_k(i,j)$ update process (so that the next merge to be taken is obvious immediately after finishing the update step).

Efficient Implementation

Data Structures: (N - # of bigrams in data [fixed])

- Hist(k) - history of merges
  Hist(k) = (a,b) merged when the remaining number of classes was k
- $c_k(i,j)$ - bigram class counts [updated after merge]
- $c_{kl}(i)$, $c_{kr}(i)$ - unigram (marginal) counts [updated]
- $L_k(a,b)$ - table of losses; upper-right triangle [updated]
- $s_k(a)$ - "subtraction" subterms [optionally updated]
- $q_k(i,j)$ - subterms involving a log [optionally updated]

The optionally updated data structures will give linear improvement only in the subsequent steps, but at least $s_k(i)$ is necessary in the initialization phase (1st iteration)

Implementation: the Initialization Phase

1. Read data in, set $k=|V|$, init counts $c_k(l,r)$; then $\forall l,r,a,b;\ a < b$:
2. Init unigram counts $c_{kl}(l)$, $c_{kr}(r)$:

$$c_{kl}(l) = \Sigma_{r=1..k} c_k(l,r), \qquad c_{kr}(r) = \Sigma_{l=1..k} c_k(l,r)$$

[must take care of start & end of data!]

3. Init $q_k(l,r)$: use the 2nd formula (count-based) on slide 27,

$$q_k(l,r) = c_k(l,r)/N \log(N\, c_k(l,r)/(c_{kl}(l)\, c_{kr}(r)))$$

4. Init $s_k(a) = \Sigma_{l=1..k} q_k(l,a) + \Sigma_{r=1..k} q_k(a,r) - q_k(a,a)$
5. Init $L_k(a,b) = s_k(a) + s_k(b) - q_k(a,b) - q_k(b,a) - q_k(a+b,a+b) +$

$$- \Sigma_{l=1..k,l\neq a,b} q_k(l,a+b) - \Sigma_{r=1..k,r\neq a,b} q_k(a+b,r)$$

Implementation: Select & Update

6. Select the best pair $\underline{(a,b)}$ to merge into $\underline{a}$

    watch the candidates when computing $L_k(a,b)$); save to $Hist(k)$

7. Optionally, update $q_k(i,j)$ for all $i,j \neq b$, get $q_{k-1}(i,j)$

    remember those $q_k(i,j)$ values needed for the updates below

8. Optionally, update $s_k(i)$ for all $i \neq b$, to get $s_{k-1}(i)$

    again, remember the $s_k(i)$ values for the “loss table” update

9. Update the loss table, $L_k(i,j)$, to $L_{k-1}(i,j)$, using the tabulated q_k, q_{k-1}, s_k and s_{k-1} values, or compute the needed $q_k(i,j)$ and $q_{k-1}(i,j)$ values dynamically from the counts:

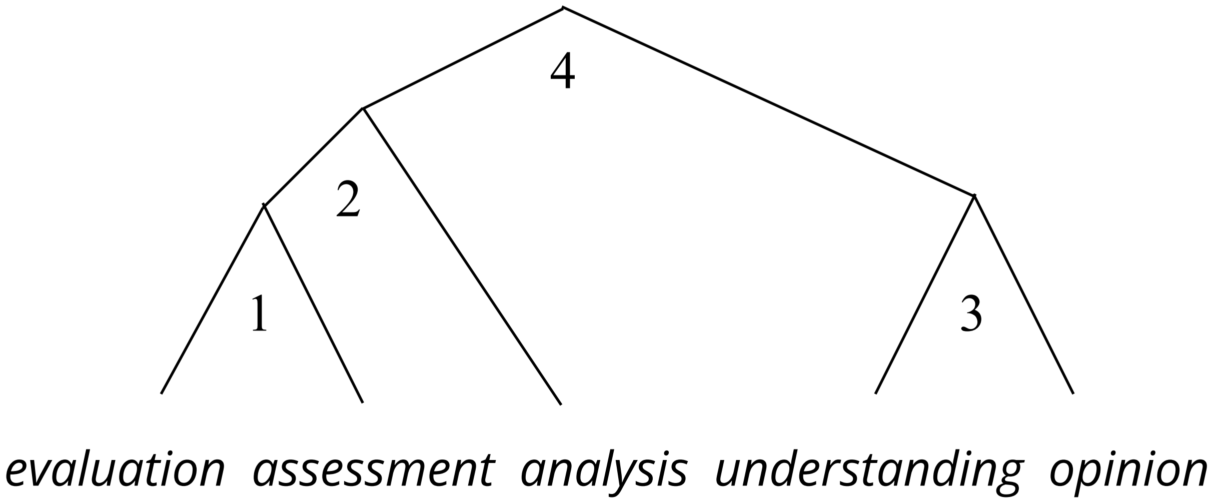
    $$c_k(i+j,b) = c_k(i,b) + c_k(j,b); \quad c_{k-1}(a,i) = c_k(a+b,i)$$

Towards the Next Iteration

10. During the $L_k(i,j)$ update, keep track of the minimal loss of MI, and the two classes which caused it.
11. Remember such best merge in Hist(k).
12. Get rid of all s_k, q_k, L_k values.
13. Set $k = k - 1$; stop if $k = 1$.
14. Start the next iteration
    - either by the optional updates (steps 7 and 8), or
    - directly updating Lk(i,j) again (step 9).

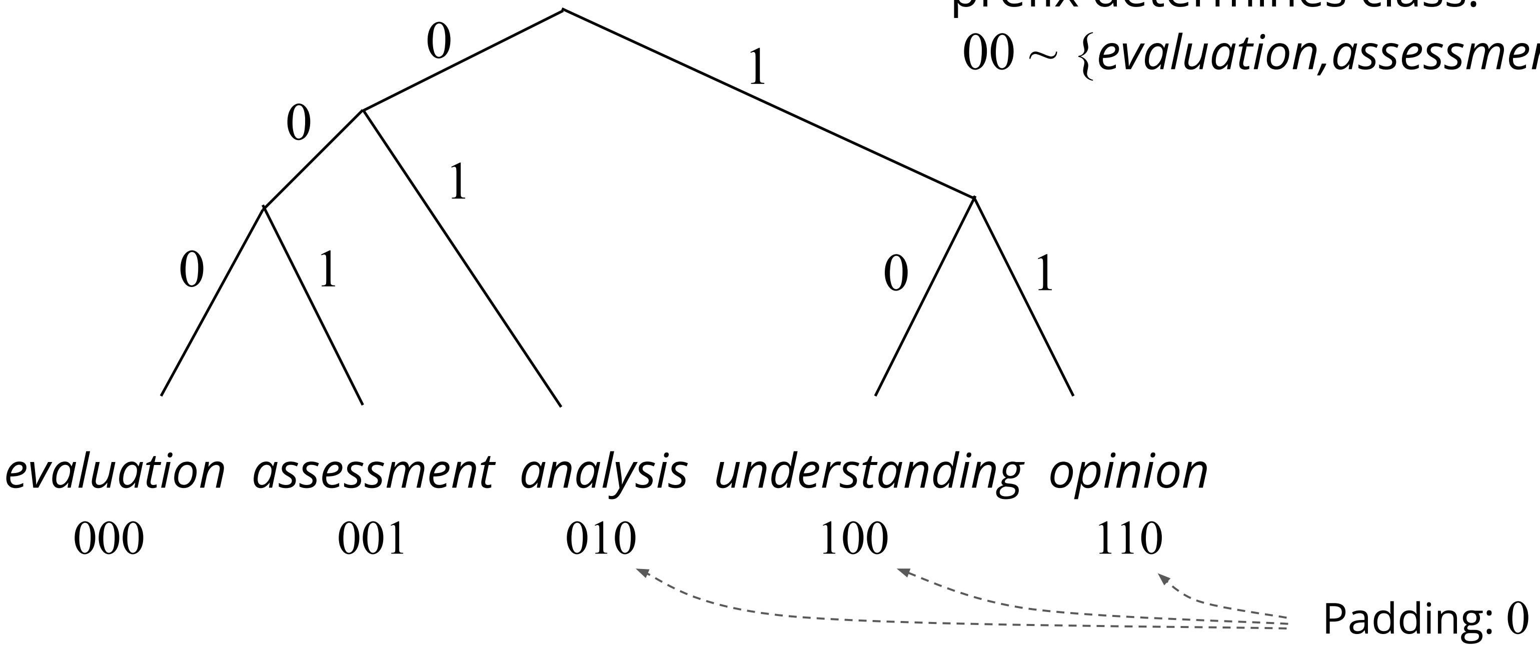
Using the Hierarchy

- Natural Form of Classes
- follows from the sequence of merges:



Numbering the Classes (within the Hierarchy)

- Binary branching
- Assign 0/1 to the left/right branch at every node:



- prefix determines class:
$00 \sim \{$*evaluation,assessment*$\}$

Word Classes in Applications

- Even in the era of neural embeddings, Brown classes have modern, practical applications (such as POS tagging)
- Especially useful in low-resource and domain-specific scenarios (tens of millions words)
- Provide compact, interpretable word classes that can replace sparse one-hot or n-gram features.

Moodle Quiz

Moodle Quiz

https://dl1.cuni.cz/course/view.php?id=18547