NPFL147

Statistical Methods in Natural Language Processing

3. Statistical language modeling

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Course Segments

1. Introduction, probability, essential information theory
2. Statistical language modelling (n-gram)
3. Statistical properties of words
4. Word embeddings
5. Hidden Markov models, Tagging

Recap from Last Week

Entropy

- Let $p_X(x)$ be a distribution of random variable X
- Basic outcomes (alphabet) Ω

$$\boxed{H(X) = - \sum_{x \in \Omega} p(x) \log_2 p(x)}$$

- Unit: bits ($\log_e$: nats)
- Notation: $H(X) = H_p(X) = H(p) = H_X(p) = H(p_X)$

Joint Entropy and Conditional Entropy

- Two random variables: X (space Ω), Y (Ψ)
- Joint entropy:
  - no big deal: (X,Y) considered a single event:

$$H(X,Y) = - \sum_{x \in \Omega} \sum_{y \in \Psi} p(x,y) \log_2 p(x,y)$$

- Conditional entropy:

$$H(Y|X) = - \sum_{x \in \Omega} \sum_{y \in \Psi} \underline{p(x,y)} \log_2 p(y|x)$$

  - recall that $H(X) = E(\log_2(1/p_X(x)))$
  - (weighted "average", and weights are not conditional)

Kullback-Leibler Distance (Relative Entropy)

- Remember:
  - long series of experiments … c_i/T_i oscillates around some number… we can only estimate it … to get a distribution q.
- So we get a distribution q; (sample space Ω, r.v. X)
- The true distribution is, however, p (same Ω, X)

⇒ how big error are we making?

- $D(p||q)$ (the Kullback-Leibler distance):

$$D(p||q) = \sum_{x \in \Omega} \underline{p(x)} \log_2 (p(x)/q(x)) = E_p \log_2 (p(x)/q(x))$$

Mutual Information

- Rewrite the definition:
  - recall: $D(r\|s) = \sum_{v \in \Omega} r(v) \log_2 (r(v)/s(v))$;
  - substitute: $r(v) = p(x,y)$, $s(v) = p(x)p(y)$; $<v> \sim <x,y>$

$$
\begin{aligned}
I(X,Y) &= D(p(x,y) \,\|\, p(x)p(y)) = \\
&= \sum_{x \in \Omega} \sum_{y \in \Psi} p(x,y) \log_2 (p(x,y)/p(x)p(y))
\end{aligned}
$$

- Measured in bits (what else? :-)

Information Inequality

$$\boxed{D(p||q) \geq 0}$$

Proof:

$$0 = -\log 1 = -\log \sum_{x\in\Omega} q(x) = -\log \sum_{x\in\Omega} p(x)(q(x)/p(x) \leq$$

...apply Jensen's inequality here ($\underline{-\log}$ is convex)...

$$\leq \sum_{x\in\Omega} p(x)(-\log(q(x)/p(x))) = \sum_{x\in\Omega} p(x)\log(p(x)/q(x)) = D(p||q)$$

Cross Entropy: The Formula

- $H_{p'}(\tilde{p}) = H(p') + D(p' \| \tilde{p})$

$$\boxed{H_{p'}(\tilde{p}) = - \sum_{x \in \Omega} p'(x) \log_2 \tilde{p}(x)}$$

- p' is certainly not the true p, but we can consider it the "real world" distribution against which we test
- note on notation (confusing): $p/p' \leftrightarrow \tilde{p}$, also $H_{T'}(p)$
- (Cross) Perplexity:

$$G_{p'}(\tilde{p}) = G_{T'}(\tilde{p}) = 2^{H_{p'}(\tilde{p})}$$

Sample Space vs. Data

- In practice, it is often inconvenient to sum over the sample space(s) Ψ, Ω (especially for cross entropy!)
- Use the following formula:

$$\boxed{\begin{aligned} H_{p'}(p) = \quad & - \sum_{y \in \Psi,\, x \in \Omega} p'(y,x) \log_2 p(y|x) = \\ & - 1/|T'| \sum_{i = 1 \ldots |T'|} \log_2 p(y_i|x_i) \end{aligned}}$$

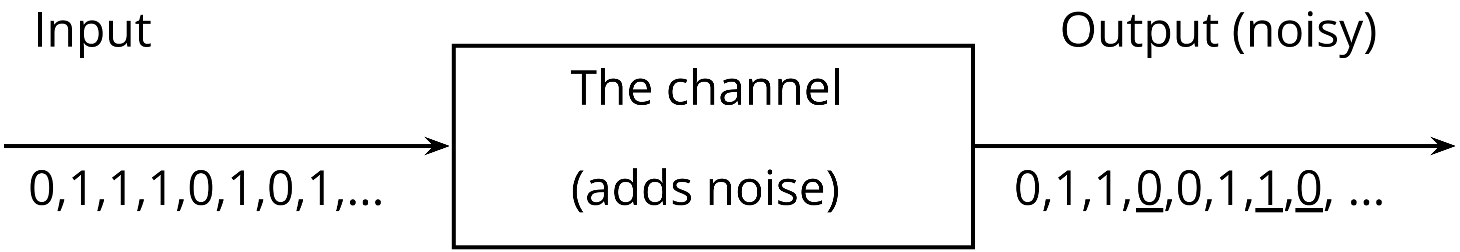
- This is the normalized log probability of the “test” data:

$$H_{p'}(p) = - 1/|T'| \log_2 \prod_{i = 1 \ldots |T'|} p(y_i|x_i)$$

Noisy Channel Model

Noisy Channel

Prototypical case:



Noisy Channel

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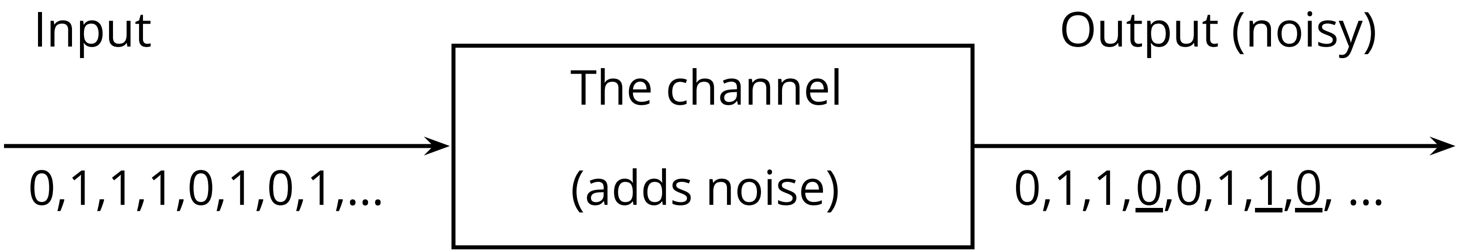

- Model: probability of error (noise)
- Example: $p(0|1) = 0.25, \quad p(1|1) = 0.75, \quad p(1|0) = 0.5, \quad p(0|0) = 0.5$
- The Task:
    - known: the noisy output; want to know: the input (decoding)

Noisy Channel

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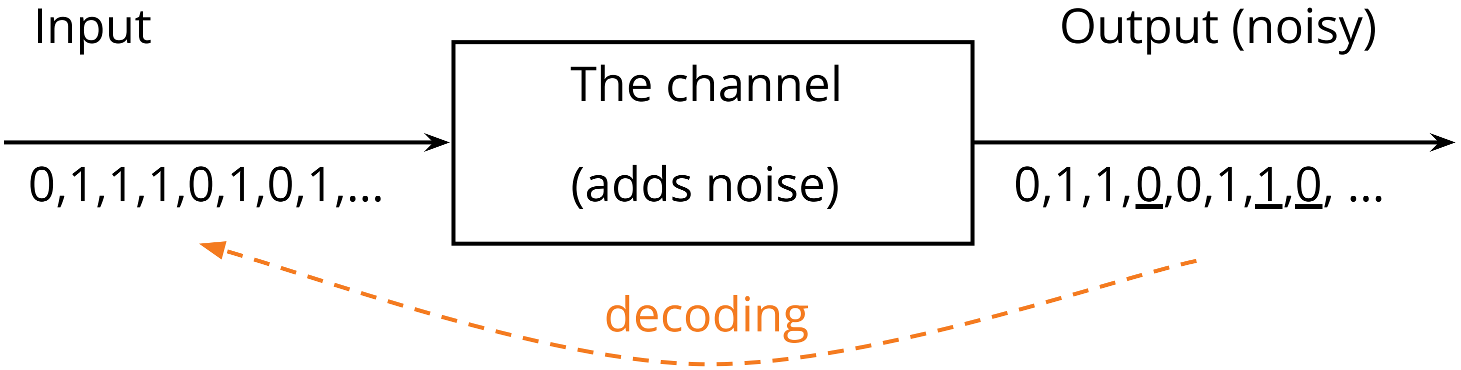

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- Example: $p(0|1) = 0.25, \quad p(1|1) = 0.75, \quad p(1|0) = 0.5, \quad p(0|0) = 0.5$
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  - known: the noisy output; want to know: the input (<u>decoding</u>)

Noisy Channel Applications

- Optical Character Recognition (OCR)
  - text → *print (adds noise), scan* → image
- Handwriting recognition (HR)
  - text → *neurons, muscles (“noise”), scan/digitize* → image
- Speech recognition (ASR)
  - text → *conversion to acoustic signal (“noise”)* → acoustic waves
- Machine Translation (MT)
  - text in target language → *translation (“noise”)* → text in source language
- Also: Part of Speech Tagging
  - sequence of tags → *selection of word forms* → text

Noisy Channel: The Golden Rule of ... OCR, HR, ASR, MT, ...

- Recall:

$$A_{best} = \text{argmax}_A p(A|B)$$

$$A_{best} = \text{argmax}_A p(B|A)\ \ p(A) / \cancel{p(B)} \qquad \textit{(Bayes Formula)}$$

$$\boxed{A_{best} = \text{argmax}_A\ p(B|A)\ p(A)} \qquad \textit{(The Golden Rule)}$$

- Where:
  - $p(B|A)$: the acoustic/image/translation/lexical model
    - application-specific name
    - will explore later
  - $p(A)$: ***<u>the language model</u>***

The Perfect Language Model

- Sequence of word forms
- Notation: $A \sim W = (w_1,w_2,w_3,\ldots,w_d)$
- The big (modeling) question:

$$p(W) = ?$$

- Well, we know (Bayes/chain rule $\rightarrow$):

$$p(W) = p(w_1,w_2,w_3,\ldots,w_d) =$$

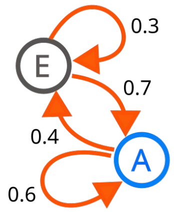
$$= p(w_1) \times p(w_2|w_1) \times p(w_3|w_1,w_2) \times\ldots\times p(w_d|w_1,w_2,\ldots,w_{d-1})$$

- Not practical (even short $W \rightarrow$ too many parameters)

Markov Chain

- Unlimited memory (cf. previous slide):
  - for w_i, we know all its predecessors $w_1, w_2, w_3, \ldots, w_{i-1}$
- Limited memory:
  - we disregard "too old" predecessors
  - remember only k previous words: $w_{i-k}, w_{i-k+1}, \ldots, w_{i-1}$
  - called "kth order Markov approximation"
- + stationary character (no change over time) → Markov Chain:

$$p(W) \cong \Pi_{i=1..d} p(w_i | w_{i-k}, w_{i-k+1}, \ldots, w_{i-1}),\ d = |W|$$



Picture from Wikipedia

n-gram Language Models

n-gram Language Model

- $(n\text{-}1)^{th}$ order Markov approximation → *n-gram Language Model*:

$$p(W) =^{df} \prod_{i=1..d} p(\overbrace{w_i}^{\text{prediction}} | \overbrace{w_{i-n+1}, w_{i-n+2}, \ldots, w_{i-1}}^{\text{history}})$$

n-gram Language Model

- $(n\text{-}1)^{th}$ order Markov approximation → *n-gram Language Model*:

$$p(W) =^{df} \prod_{i=1..d} p(\overset{\text{prediction}}{w_i} | \overbrace{w_{i-n+1}, w_{i-n+2}, \ldots, w_{i-1}}^{\text{history}})$$

- In particular (assume vocabulary $|V| = 60k$):
  - 0-gram LM: uniform model, $p(w) = 1/|V|$, 1 parameter
  - 1-gram LM: unigram model, $p(w)$, 6×10^4 parameters
  - 2-gram LM: bigram model, $p(w_i|w_{i-1})$, 3.6×10^9 parameters
  - 3-gram LM: trigram model, $p(w_i|w_{i-2},w_{i-1})$, 2.16×10^{14} parameters

LM: Observations

- How large n?
  - Nothing is enough (theoretically)
  - But anyway: as much as possible ($\rightarrow$ close to “perfect” model)
  - Neural models allow context/history of thousands of words
  - n-gram models: 3–7
    - parameter estimation? (*reliability, data availability, storage space, …*)
    - 4 is too much: $|V|=60k \rightarrow 1.296 \times 10^{19}$ parameters
    - but: 6-7 would be (almost) ideal (having enough data)
    - in fact: one can recover the original text sequence from 7-grams!
- Reliability ~ $(1 / \text{Detail})$ ($\rightarrow$ need compromise)

The Length Issue

- $\forall n: \Sigma_{w \in \Omega^n} p(w) = 1 \Rightarrow \Sigma_{n=1..\infty} \Sigma_{w \in \Omega^n} p(w) >> 1 \ (\rightarrow \infty)$
- We want to model all sequences of words
  - for “fixed” length tasks: no problem - n fixed, sum is 1
    - *tagging, OCR/handwriting (if words identified ahead of time)*
  - for “variable” length tasks: have to account for
    - *discount shorter sentences*

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    - *discount shorter sentences*
- General model:
  - for each sequence of words of length n, define:

    $p'(w) = \lambda_n p(w)$ such that $\Sigma_{n=1..\infty} \lambda_n = 1 \Rightarrow \Sigma_{n=1..\infty} \Sigma_{w \in \Omega^n} p'(w) = 1$

  - e.g., estimate λ_n from data; or use normal or other distribution

Parameter Estimation

- Parameter: numerical value needed to compute $p(w|h)$
- From data (how else?)

- Data preparation:
  - get rid of formatting etc. (“text cleaning”)
  - define words (separate but include punctuation, call it “word”)
  - define sentence boundaries (insert “words” <s> and </s>)
  - letter case: keep, discard, or be smart:
    - name recognition
    - number type identification
    - [these are huge problems per se!]

  - numbers: keep, replace by <num>, or be smart (form ~ pronunciation)

Maximum Likelihood Estimate

- MLE: Relative Frequency ...
  - ...best predicts the data at hand (the "training data")
- Trigrams from Training Data T:
  - count sequences of three words in T: $c_3(w_{i-2},w_{i-1},w_i)$
    (*notation: just saying that the three words follow each other*)
  - count sequences of two words in T: $c_2(w_{i-1},w_i)$:
    - either use $\mathbf{c_2(y,z) = \Sigma_w\ c_3(y,z,w)}$
    - or count differently at the beginning (& end) of data!

$$\boxed{p(w_i|w_{i-2},w_{i-1}) =^{est} c_3(w_{i-2},w_{i-1},w_i) \,/\, c_2(w_{i-2},w_{i-1})}$$

Character Language Model

- Use individual characters instead of words:

$$p(W) =^{df} \prod_{i=1..d} p(c_i|c_{i-n+1},c_{i-n+2},\ldots,c_{i-1})$$

- Same formulas etc.
- Might consider 4-grams, 5-grams or even more
- Good for language comparison (how?)
- Transform cross-entropy between letter- and word-based models:

$$H_S(p_{char}) = H_S(p_{word}) \;/\; \text{avg. \# of characters per word in } S$$

LM: an Example

- Training data: <s> <s> He can buy the can of soda .

LM: an Example

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- <u>Unigram model (n=8):</u>
    - $p_1(\text{He}) = p_1(\text{buy}) = p_1(\text{the}) = p_1(\text{of}) = p_1(\text{soda}) = p_1(.) = 0.125$,
      $p_1(\text{can}) = 0.25$

LM: an Example

- Training data: <s> <s> He can buy the can of soda .
- Unigram model (n=8):
  - $p_1(\text{He}) = p_1(\text{buy}) = p_1(\text{the}) = p_1(\text{of}) = p_1(\text{soda}) = p_1(.) = 0.125$, $p_1(\text{can}) = 0.25$
- Bigram model:
  - $p_2(\text{He}|\text{<s>}) = 1$, $p_2(\text{can}|\text{He}) = 1$, $p_2(\text{buy}|\text{can}) = 0.5$, $p_2(\text{of}|\text{can}) = 0.5$, $p_2(\text{the}|\text{buy}) = 1, \ldots$

LM: an Example

- Training data: $\langle s \rangle \langle s \rangle$ He can buy the can of soda .
- Unigram model (n=8):
  - p_1(He) = p_1(buy) = p_1(the) = p_1(of) = p_1(soda) = p_1(.) = 0.125, p_1(can) = 0.25
- Bigram model:
  - $p_2(\text{He}|\langle s \rangle) = 1$, $p_2(\text{can}|\text{He}) = 1$, $p_2(\text{buy}|\text{can}) = 0.5$, $p_2(\text{of}|\text{can}) = 0.5$, $p_2(\text{the}|\text{buy}) = 1$,...
- Trigram model:
  - $p_3(\text{He}|\langle s \rangle,\langle s \rangle) = 1$, $p_3(\text{can}|\langle s \rangle,\text{He}) = 1$, $p_3(\text{buy}|\text{He,can}) = 1$, $p_3(\text{of}|\text{the,can}) = 1$, ..., $p_3(.|\text{of,soda}) = 1$

LM: an Example

- Training data: <s> <s> He can buy the can of soda .
- Unigram model (n=8):
  - $p_1(\text{He}) = p_1(\text{buy}) = p_1(\text{the}) = p_1(\text{of}) = p_1(\text{soda}) = p_1(.) = 0.125$, $p_1(\text{can}) = 0.25$
- Bigram model:
  - $p_2(\text{He}|\text{<s>}) = 1$, $p_2(\text{can}|\text{He}) = 1$, $p_2(\text{buy}|\text{can}) = 0.5$, $p_2(\text{of}|\text{can}) = 0.5$, $p_2(\text{the}|\text{buy}) = 1, \ldots$
- Trigram model:
  - $p_3(\text{He}|\text{<s>},\text{<s>}) = 1$, $p_3(\text{can}|\text{<s>},\text{He}) = 1$, $p_3(\text{buy}|\text{He},\text{can}) = 1$, $p_3(\text{of}|\text{the},\text{can}) = 1, \ldots, p_3(.|\text{of},\text{soda}) = 1$
- Entropy: $H(p_1) = 2.75$, $H(p_2) = 0.25$, $H(p_3) = 0$ ← *Great?!*

LM: an Example (The Problem)

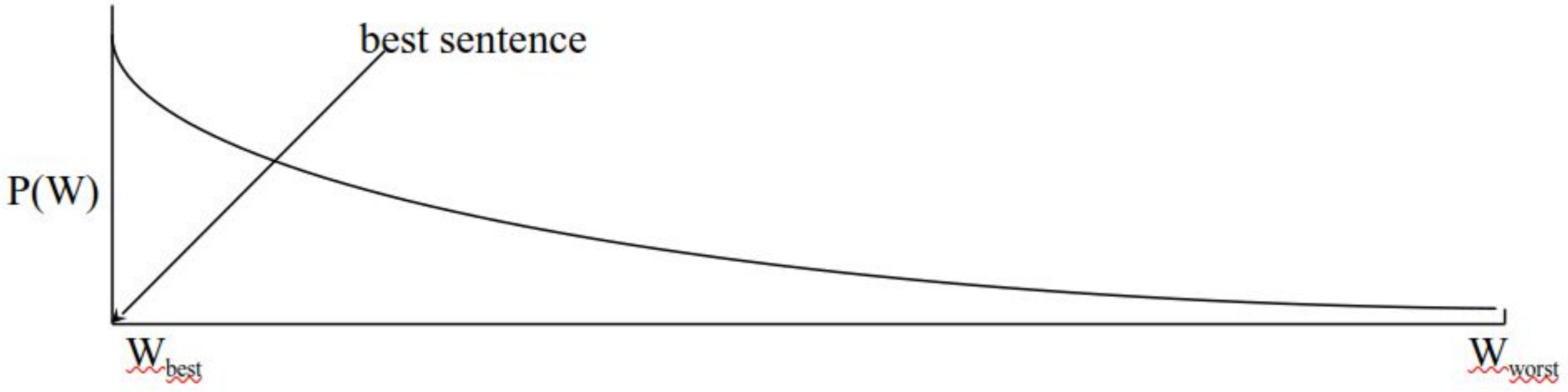
- Test data:

$$S = \text{<s> <s> It was the greatest buy of all.}$$

- Cross entropy:
    - Even $H_S(p_1)$ fails ($= H_S(p_2) = H_S(p_3) = \infty$), because:
    - all unigrams but $p_1(\text{the})$, $p_1(\text{buy})$, $p_1(\text{of})$ and $p_1(.)$ are 0.
    - all bigram probabilities are 0.
    - all trigram probabilities are 0.
- We want:
    - to make all (theoretically possible) probabilities non-zero.

Real World Situation (from Lecture 1)

- Unable to specify set of grammatical sentences today using fixed "categorical" rules (maybe never, cf. arguments in M&S)
- Use statistical "model" based on **real world data** and care about the best sentence only (disregarding the "grammaticality" issue)



Moodle Quiz

Moodle Quiz

https://dl1.cuni.cz/course/view.php?id=18547