NTIN066 - solutions 3

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1. Prove that a naive splay strategy using only single rotations cannot have better amortized complexity than $\Omega(n)$.
Consider what happens when splaying a path.

When we sequentially splay all the nodes of a path (from the lowest one), the shapes of the intermediate trees are always two paths going from the root. One path is formed from already accessed keys, the other is formed from not-yet accessed keys. Therefore, the average depth of the node splayed is approximately $n/2$ and so the amortized complexity cannot be better than $\Omega(n)$.

2. What happens if we splay all the nodes of the path (degenerated tree) from the lowest to the highest one?

The shape of the resulting tree will be also a path. However, the shapes of the intermediate trees are much better balanced. So the amortized complexity of a sequential splay of a very long sequence is good. Try it here: `https://www.cs.usfca.edu/~galles/visualization/SplayTree.html`

3. What is the potential of a path and of a perfectly balanced tree?
Can we reach the limits $0 \leq \Phi \leq n\log(n)$?

For the path, the potential is:

$$\Phi = \log(1) + \log(2) + \ldots + \log(n) = \sum_{i=1}^{n} \log(i) \leq n(\log(n))$$

For a perfectly balanced tree with $n = 2^k - 1$ nodes:

- *The potential of of the root:* $\log(2^k - 1) < k$
- *The potential of the root's children:* $2 \cdot \log(2^{k-1} - 1) < 2(k-1)$
- *The potential of the root's grandchildren:* $4 \cdot \log(2^{k-2} - 1) < 4(k-2)$
- $\ldots$
- *The potential of all leaves together:* $2^{k-1} \cdot log(2^{k-(k-1)} - 1) < 2^{k-1}(k - (k-1))$

The potential of the whole tree is $\Phi < \sum_{i=0}^{k-1} 2^i(k-i)$. *After substituting* $i = k-j$, *we get:*

$$\Phi < \sum_{j=1}^{k} 2^{k-j} j = \sum_{j=1}^{k} 2^k \frac{j}{2^j} = (n+1) \sum_{j=1}^{k} \frac{j}{2^j} < (n+1) \sum_{j=1}^{\infty} \frac{j}{2^j} = 2n+2$$

The last equation comes from the following enumeration:
$\sum_{j=1}^{\infty} \frac{j}{2^j} =$
$\frac{1}{2} +$
$\frac{1}{4} + \frac{1}{4} +$
$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} +$
$\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} +$
$\cdots$
$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$
Hence the potential of a perfectly balanced tree is linear with the number of nodes.