Dirichlet-Categorical distributions

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Many of the slides in this presentation were taken from the presentations of Carl Edward Rasmussen (University of Cambridge)

Multinomial distribution

Generalisation of the binomial distribution from 2 outcomes to m outcomes.
Useful for random variables that take one of a finite set of possible outcomes.
Throw a die $n = 60$ times, and count the observed (6 possible) outcomes.

Outcome	Count
$X = x_1 = 1$	$k_1 = 12$
$X = x_2 = 2$	$k_2 = 7$
$X = x_3 = 3$	$k_3 = 11$
$X = x_4 = 4$	$k_4 = 8$
$X = x_5 = 5$	$k_5 = 9$
$X = x_6 = 6$	$k_6 = 13$

Note that we one parameter here is redundant. We don't need to know all the k_i and n, because $\sum_{i=1}^{6} k_i = n$.

Multinomial Distribution

Consider a discrete random variable X that can take one of m values $x_1, \dots, x_m$.
Out of n independent trials, let k_i be the number of times $X = x_i$ was observed.
It follows that $\sum_{i=1}^{m} k_i = n$.
Denote by π_i the probability that $X = x_i$, with $\sum_{i=1}^{m} \pi_i = 1$.
The probability of observing a vector of occurrences $k = [k_1, \dots, k_m]$ is given by the multinomial distribution parametrized by $\pi = [\pi_1, \dots, \pi_m]$.

$$p(\vec{k}|\vec{\pi}, n) = p(k_1, \dots, k_m|\pi_1, \dots, \pi_m, n) = \frac{n!}{k_1!k_2!\dots k_m!} \prod_{i=1}^{m} \pi_i^{k_i}$$

- Note that we can write $p(\vec{k}|\vec{\pi})$ since n is redundant.
- The multinomial coefficient $\frac{n!}{k_1!k_2!\dots k_m!}$ is a generalisation of the binomial coefficient $\binom{n}{k}$.

The *Categorical* distribution is the generalisation of the *Bernoulli* distribution to m outcomes, and the special case of the *multinomial* distribution with one trial:

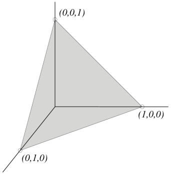
$$p(X = x_i|\vec{\pi}) = \pi_i$$

Dirichlet Distribution

The *Dirichlet* distribution is to the *Categorical/Multinomial* what the *Beta* distribution is to the *Bernoulli/Binomial*.

It is a generalisation of the Beta defined on the $m-1$ dimensional simplex.

- Consider the vector $\vec{\pi} = [\pi_1, \dots, \pi_m]$, with $\sum_{i=1}^m \pi_i = 1$ and $\pi_i \in (0,1)\ \forall i$.
- The vector $\vec{\pi}$ lives in the open standard $m-1$ simplex.
- $\vec{\pi}$ could be a parameter vector of a multinomial distribution.

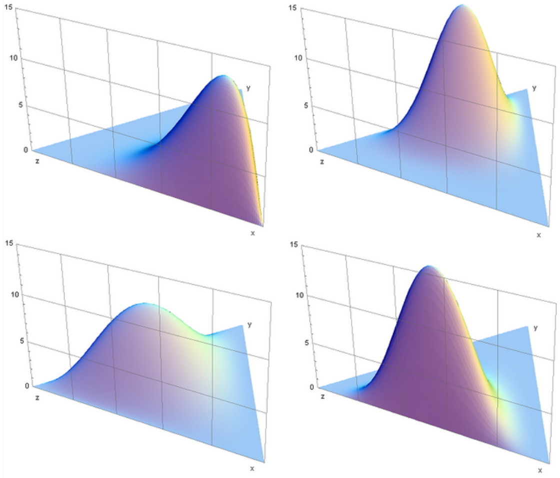


The Dirichlet distribution is given by

$$Dir(\vec{\pi}|\alpha_1, \dots, \alpha_m) = \frac{\Gamma(\sum_{i=1}^m \alpha_i)}{\prod_{i=1}^m \Gamma(\alpha_i)} \prod_{i=1}^m \pi_i^{\alpha_i - 1} = \frac{1}{B(\vec{\alpha})} \prod_{i=1}^m \pi_i^{\alpha_i - 1}$$

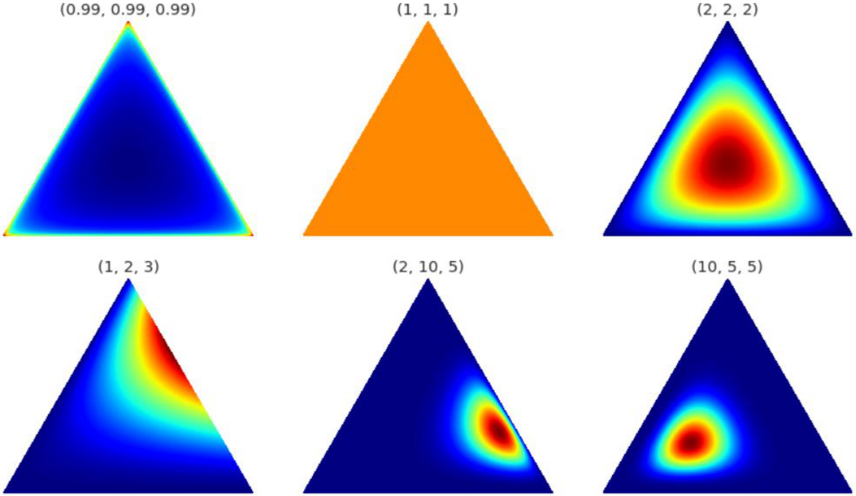
- $\vec{\alpha} = [\alpha_1, \dots, \alpha_m], \quad \forall i : \alpha_i > 0$
- $B(\vec{\alpha})$ is the multivariate beta function (normalization)
- $E(\pi_j) = \frac{\alpha_j}{\sum_{i=1}^m \alpha_i}$ is the mean for the j-th element.

Dirichlet Distribution: examples



Several images of the probability density of the Dirichlet distribution when $n = 3$ for various parameter vectors $\vec{\alpha}$. Clockwise from top left: $\vec{\alpha} = (6, 2, 2), (3, 7, 5), (6, 2, 6), (2, 3, 4)$.

Dirichlet Distribution: examples

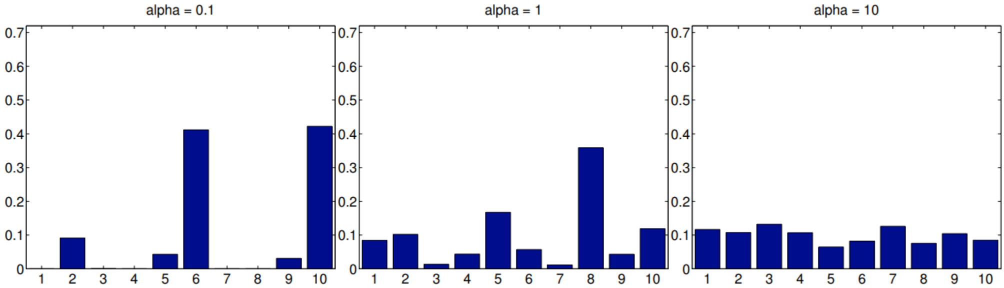


The symmetric Dirichlet distribution

In the symmetric Dirichlet distribution, all parameters are identical: $\alpha_i = \alpha, \ \forall i$.

`en.wikipedia.org/wiki/File:LogDirichletDensity-alpha_0.3_to_alpha_2.0.gif`

Distributions drawn at random from symmetric 10 dimensional Dirichlet distributions with various concentration parameters.



Beta and Dirichlet: definition

Beta distribution:

$$Beta(\pi|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\pi^{\alpha-1}(1-\pi)^{\beta-1}$$

Dirichlet distribution (generalization of Beta distribution to m outcomes):

$$Dir(\vec{\pi}|\alpha_1,\ldots,\alpha_m) = \frac{\Gamma(\sum_{i=1}^{m}\alpha_i)}{\prod_{i=1}^{m}\Gamma(\alpha_i)}\prod_{i=1}^{m}\pi_i^{\alpha_i-1}$$

Beta and Dirichlet: posterior probability

Posterior probability for Beta-Bernoulli distribution:

$$p(\pi|D) = \frac{p(\pi|\alpha,\beta)p(D|\pi)}{p(D)} \propto \pi^{\alpha-1}(1-\pi)^{\beta-1}\pi^{k}(1-\pi)^{n-k} = \pi^{\alpha+k-1}(1-\pi)^{\beta+n-k-1}$$

$$p(\pi|D) = Beta(\pi|\alpha+k, \beta+n-k)$$

Posterior probability for Dirichlet-Categorical distribution:

$$p(\vec{\pi}|D) = \frac{p(\vec{\pi}|\vec{\alpha})p(D|\vec{\pi})}{p(D)} \propto \prod_{i=1}^{m} \pi_i^{\alpha_i-1}\pi_i^{k_i} = \prod_{i=1}^{m} \pi_i^{\alpha_i+k_i-1} \propto Dir(\vec{\pi}|\vec{\alpha}+\vec{k})$$

Beta and Dirichlet: predictive distributions

Beta-Bernoulli predictions:

$$p(x_{next} = 1|D) = \int_0^1 p(x_{next} = 1|\pi)p(\pi|D)d\pi = \int_0^1 \pi Beta(\pi|\alpha{+}k, \beta{+}n{-}k)d\pi = \frac{\alpha + k}{\alpha + \beta + n}$$

Dirichlet-Categorical predictions:

$$p(x_{next} = j|D) = \int_\triangle p(x_{next} = j|\vec{\pi})p(\vec{\pi}|D)d\vec{\pi} = \int_\triangle \pi_j Dir(\vec{\pi}|\vec{\alpha} + \vec{k})d\vec{\pi} = \frac{\alpha_j + k_j}{\sum_{i=1}^m (\alpha_i + k_i)}$$

The sign $\triangle$ indicates a simplex: the integral is taken across all vectors $\vec{\pi}$ that are valid probability distributions, i.e. $\sum_{i=1}^m \pi_i = 1$.

The integrals are equal to expected values of given Beta/Dirichlet posterior distributions.