


Beta-Bernoulli distributions

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unless otherwise stated

Many of the slides in this presentation were taken from the presentations of Carl Edward Rasmussen (University of Cambridge)

Coin tossing

You are presented with a coin.



What is the probability of heads?

How is the probability defined?

We need data!

We toss once and it's head (H).

How much are you willing to bet $p(H) > 0.5$?

Bernoulli distribution

The Bernoulli probability distribution over binary random variables:

- Binary random variable X : outcome x of a single coin toss.
- x can take two values: $X = 0$ for tails and $X = 1$ for heads.
- Let the probability of heads be $\pi = p(X = 1)$.
- π is the parameter of the Bernoulli distribution.
- The probability of tail is $p(X = 0) = 1 - \pi$. We can compactly write

$$p(X = x|\pi) = p(x|\pi) = \pi^x(1 - \pi)^{1-x}$$

What do we think π is after observing a single head outcome “H”?

- Maximum likelihood! We maximise the probability of data with respect to the parameter π :

$$p(H|\pi) = p(x = 1|\pi) = \pi, \quad \operatorname{argmax}_{\pi \in [0,1]} \pi = 1$$

- Ok, so the answer is $\pi = 1$. This coin only generates heads.

Is this reasonable? How much are you willing to bet $p(H) > 0.5$?

Binomial distribution

We observe a sequence of tosses rather than a single toss: HHTH

- The probability of this particular sequence is: $p(\text{HHTH}) = \pi^3(1 - \pi)$.
- But so is the probability of THHH, of HTHH and of HHHT.
- We often don't care about the order of the outcomes, only about the counts.

In our example, the probability of 3 heads out of 4 tosses is: $4\pi^3(1 - \pi)$.

The binomial distribution gives the probability of observing k heads out of n tosses

$$p(k|\pi, n) = \binom{n}{k} \pi^k (1 - \pi)^{n-k}$$

- This assumes n independent tosses from a Bernoulli distribution $p(x|\pi)$.
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient, also known as “n choose k”

Maximum likelihood estimation

If we observe k heads out of n tosses, what do we think π is?

We can maximise the likelihood of the observed data given the parameter π .

$$p(k|\pi, n) \propto \pi^k (1 - \pi)^{n-k}$$

It is convenient to take the logarithm and derivatives with respect to π :

$$\log p(k|\pi, n) = k \log \pi + (n - k) \log(1 - \pi) + \text{Constant}$$

$$\frac{\partial \log p(k|\pi, n)}{\partial \pi} = \frac{k}{\pi} - \frac{n - k}{1 - \pi} = 0 \iff \pi = \frac{k}{n}$$

Is this reasonable?

- For HHTH we get $\pi = 3/4$.

How much would you bet now that $p(\text{H}) > 0.5$?

We would need a probability over a probability...

Prior beliefs about coins

So we have observed 3 heads out of 4 tosses but are unwilling to bet much money that $p(\text{H}) > 0.5$? (That for example out of 10,000,000 tosses at least 5,000,001 will be heads.)

Why?

- You might believe that coins tend to be fair. $\pi \simeq 1/2$
- A finite set of observations updates your opinion about π .
- But how to express your opinion about π before you see any data?

Pseudo-counts: You think the coin is fair and... you are...

- Not very sure. You act as if you had seen 2 heads and 2 tails before.
- Pretty sure. It is as if you had observed 20 heads and 20 tails before.
- Totally sure. As if you had seen 1000 heads and 1000 tails before.

Depending on the strength of your prior assumptions, it takes a different number of actual observations to change your mind.

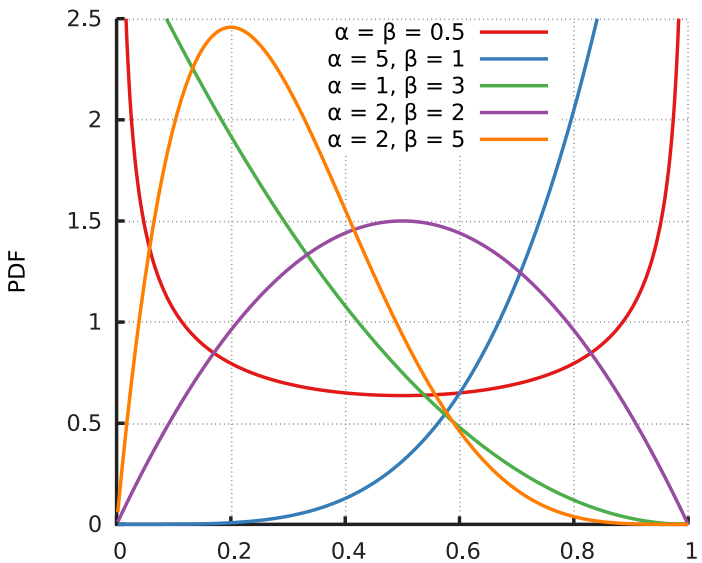
Beta distribution: distributions on probabilities

Continuous probability distribution defined on the interval $[0, 1]$

$$\text{Beta}(\pi|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \pi^{\alpha-1} (1 - \pi)^{\beta-1} = \frac{1}{B(\alpha, \beta)} \pi^{\alpha-1} (1 - \pi)^{\beta-1}$$

- $\alpha > 0$ and $\beta > 0$ are the shape parameters.
- these parameters correspond to 'one plus the pseudo-counts'.
- $\Gamma(\alpha)$ is an extension of the factorial function.
(https://en.wikipedia.org/wiki/Gamma_function)
- $\Gamma(n) = (n - 1)!$ for integer n and $\Gamma(n + 1) = n\Gamma(n)$
- $B(\alpha, \beta) = \int_0^1 \pi^{\alpha-1} (1 - \pi)^{\beta-1} d\pi$ is the normalization function so that it sums up to one.
- The mean is given by $E(\pi) = \frac{\alpha}{\alpha + \beta}$

Beta distribution: examples



Beta distribution: online demo

<https://mathlets.org/mathlets/beta-distribution>

Beta distribution: exercise

Could you imagine the shape of coin that would fit the following parameters for the prior Beta distributions?

- $\alpha = 100, \beta = 100$
- $\alpha = 2, \beta = 3$
- $\alpha = 2, \beta = 10$
- $\alpha = 0.1, \beta = 0.1$
- $\alpha = 1, \beta = 1$

Posterior \propto Prior \times Likelihood

Imagine we observe k heads out of n tosses.

The probability of the observed data given π is the likelihood:

$$p(D|\pi) = p(k|n, \pi) \propto \pi^k (1 - \pi)^{n-k}$$

We use our prior $p(\pi|\alpha, \beta) = \text{Beta}(\pi|\alpha, \beta)$ to get the posterior probability (Bayes' theorem):

$$p(\pi|D) = \frac{p(\pi|\alpha, \beta)p(D|\pi)}{p(D)} \propto \pi^{\alpha-1} (1 - \pi)^{\beta-1} \pi^k (1 - \pi)^{n-k} = \pi^{\alpha+k-1} (1 - \pi)^{\beta+n-k-1}$$

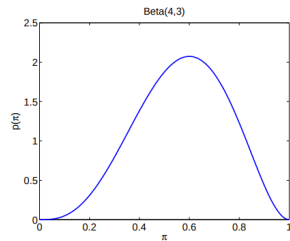
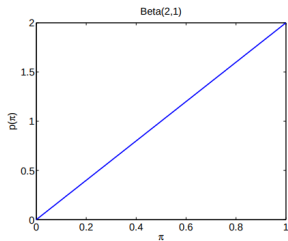
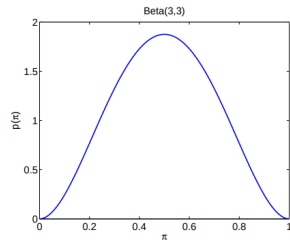
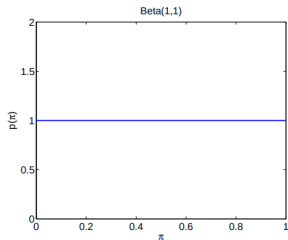
$$p(\pi|D) = \text{Beta}(\pi|\alpha + k, \beta + n - k)$$

The Beta distribution is a *conjugate* prior to the Bernoulli/binomial distribution:

- The resulting posterior is also a Beta distribution, but with different parameters.
- The posterior parameters are: $\alpha_{\text{posterior}} = \alpha_{\text{prior}} + k$, $\beta_{\text{posterior}} = \beta_{\text{prior}} + (n - k)$

Before and after observing one head

Prior



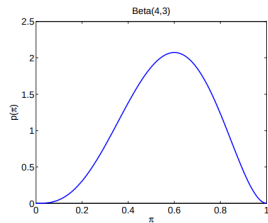
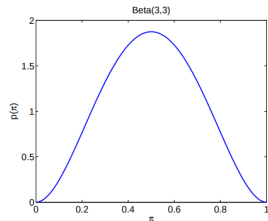
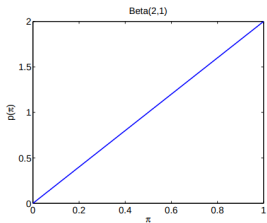
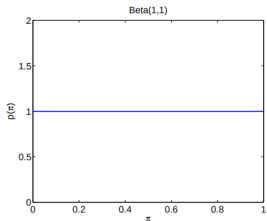
Posterior

Posterior Beta distribution: online demo

<http://www.randomservices.org/random/apps/BetaCoin.html>

Question

Prior



Posterior

What is the probability of $\pi = 0.5$?

Making predictions

Given some data D , what is the probability of the next toss being head, $x_{next} = 1$?

Under the **Maximum Likelihood approach** we predict using the value of π_{ML} that maximizes the likelihood of π given the observed data D :

$$p(x_{next} = 1 | \pi_{ML}) = \pi_{ML}$$

With the **Bayesian approach**, we integrate over all possible parameter settings:

$$p(x_{next} = 1) = \int_0^1 p(x_{next} = 1 | \pi) p(\pi | D) d\pi = \int_0^1 \pi \text{Beta}(\pi | \alpha + k, \beta + n - k) d\pi = \frac{\alpha + k}{\alpha + \beta + n}$$

The prediction for heads happens to correspond to the mean of the posterior distribution.

E.g., if we observe only one head:

- Learner A with $\text{Beta}(1, 1)$ predicts $p(x_{next} = 1 | D) = 2/3$
- Learner B with $\text{Beta}(3, 3)$ predicts $p(x_{next} = 1 | D) = 4/7$

Expected value of the Beta distribution

Predictive probability of heads is equal to the expected value of the posterior distribution.

Here, we derive the expected value for $X \sim \text{Beta}(\alpha, \beta)$:

$$\begin{aligned} E[X] &= \int_0^1 \pi \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \pi^{\alpha-1} (1 - \pi)^{\beta-1} d\pi \\ &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \pi^\alpha (1 - \pi)^{\beta-1} d\pi \\ &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \pi^{\alpha+1-1} (1 - \pi)^{\beta-1} d\pi \\ &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot B(\alpha + 1, \beta) \\ &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha + 1)\Gamma(\beta)}{\Gamma(\alpha + \beta + 1)} \\ &= \frac{\Gamma(\alpha + \beta) \cdot \alpha\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta) \cdot (\alpha + \beta)\Gamma(\alpha + \beta)} = \frac{\alpha}{\alpha + \beta} \end{aligned}$$

Making predictions - other statistics

Given the posterior distribution, we can also answer other questions such as “what is the probability that $\pi > 0.5$ given the observed data?”

$$p(\pi > 0.5|D) = \int_{0.5}^1 p(\pi'|D)d\pi' = \int_{0.5}^1 \text{Beta}(\pi'|\alpha', \beta')d\pi'$$

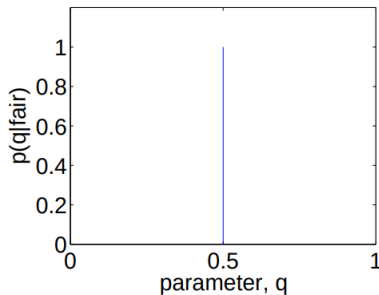
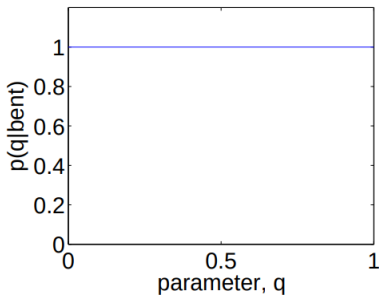
- Learner A with prior $\text{Beta}(1, 1)$ predicts $p(\pi > 0.5|D) = 0.75$
- Learner B with prior $\text{Beta}(3, 3)$ predicts $p(\pi > 0.5|D) = 0.66$

Learning about a coin, multiple models

Consider two alternative models of a coin, “fair” and “bent”. A priori, we may think that “fair” is more probable, e.g.:

$$p(\text{fair}) = 0.8, \quad p(\text{bent}) = 0.2$$

For the bent coin, (a little unrealistically) all parameter values could be equally likely, where the fair coin has a fixed probability:



Learning about a coin, multiple models

We make 10 tosses, and get data D : T H T H T T T T T T

The evidence for the fair model is: $p(D|fair) = (1/2)^{10} \simeq 0.001$ and for the bent model:

$$p(D|bent) = \int_0^1 p(D|\pi, bent)p(\pi|bent)d\pi = \int_0^1 \pi^2(1-\pi)^8d\pi = B(3, 9) \simeq 0.002$$

Using priors $p(fair) = 0.8$, $p(bent) = 0.2$, the posterior by Bayes rule:

$$p(fair|D) \propto p(fair) \cdot p(D|fair) = 0.0008,$$

$$p(bent|D) \propto p(bent) \cdot p(D|bent) = 0.0004,$$

i.e., two thirds probability that the coin is fair.

How do we make predictions? By weighting the predictions from each model by their probability. Probability of Head at next toss is:

$$\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{3}{12} = \frac{5}{12}$$

Frequentist statistics

- Predictions on the underlying truths of the experiment use only data from the current experiment.
- Maximum likelihood estimation – we maximize the probability of data.

Bayesian statistics

- Predictions take past knowledge of similar experiments into account. These are known as *prior*. This prior is combined with current experiment data to get a *posterior*.
- Maximum a posteriori estimation – we maximize the probability of parameters given the observed data and our prior knowledge.