5. Cvičení z MA II. (23. 3. 2010)

Určete primitivní funkce k následujícím funkcím:

1. Rozcvička:

(a) $\int \frac{\log x}{x(1+\log x)}\, dx$ (b) $\int \operatorname{arctg} \sqrt{x}\, dx$

(c) $\int \log(x+\sqrt{1+x^2})\, dx$ (d) $\int \arccos x\, dx$

(e) $\int \frac{\arcsin\sqrt{x}}{\sqrt{1-x}}\, dx$

2. Zvolte vhodnou substituci a spočítejte (na intervalech, které jsou 'přirozeným' definičním oborem výsledných primitivních funkcí):

(a) $\int \sin^3 x \cos^2 x\, dx$ (b) $\int \frac{\sin x}{(1-\cos x)^2}\, dx$

(c) $\int \frac{1}{1+\operatorname{tg} x}\, dx$ (d) $\int \frac{\sin^2 x}{1+\sin^2 x}\, dx$

(e) $\int \frac{1}{(\sin x+\cos x)^2}\, dx$ (f) $\int \frac{1}{2\sin x-\cos x+5}\, dx$

(g) $\int \frac{1}{5+4\sin x}\, dx$ (h) $\int \frac{1}{\sin^2 x+\operatorname{tg}^2 x}\, dx$

(i) $\int \frac{1}{(2+\cos x)\sin x}\, dx$ (j) $\int \frac{1}{\sin x\cos^4 x}\, dx$

(k) $\int \frac{1}{\sqrt{1+e^x}}\, dx$

3. Nepříjemné substituce:

(a) $\int \frac{1}{x}\sqrt{\frac{x-1}{x+1}}\, dx$ (b) $\int \frac{1}{\sqrt{x^2+1}}\, dx$

(c) $\int \frac{1}{\sqrt{x^2-1}}\, dx$ (d) $\int \frac{1}{\sqrt{x^2+x+1}}\, dx$

(e) $\int \frac{1}{x+\sqrt{x^2+x+1}}\, dx$ (f) $\int \frac{1}{x\sqrt{x^2+x+1}}\, dx$

(g) $\int \frac{1}{x\sqrt{x^2+5x+1}}\, dx$ (h) $\int \frac{1}{\sqrt{2+x-x^2}}\, dx$

4. Příklady písemkového typu (doc. Kalenda):

(a) $\int \frac{\sin^2 x}{\sin x+cos x+2}\, dx$ (b) $\int \frac{x}{x^2+7+\sqrt{x^2+7}}\, dx$ (c) $\int \frac{x^2+1}{(x-1)(x^2-1)(x^2+x+1)}\, dx$

(d) $\int \frac{(\operatorname{tg} x+\operatorname{cotg} x)^2}{\sin^2 x-\cos^2 x}\, dx$ (e) $\int \frac{\sin x}{9\cos^2 x+2\sin^4 x}\, dx$ (f) $\int \frac{\sqrt{x+1}}{\sqrt{x+1}+\sqrt{x}}\, dx$

(g) $\int \frac{\cos^2 x}{\sin x(1-\cos x)}\, dx$

Řešení: (až na c)

1a. $\log x - \log|1+\log x|$, na $(0, \frac{1}{e})$ a na $(\frac{1}{e}, \infty)$ **1b.** $(x+1)\,\mathrm{arctg}\,\sqrt{x} - \sqrt{x}$, na $(0,\infty)$ **1c.** $x\log(x+\sqrt{x^2+1}) - \sqrt{x^2+1}$, na R **1d.** $x\cdot\arccos x - \sqrt{1-x^2}$, na $(-1,1)$ **1e.** $-2\sqrt{1-x}\cdot\arcsin\sqrt{x} + 2\sqrt{x}$, na $(0,1)$

2a. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$, na R **2b.** $\frac{1}{\cos x - 1}$, na $(2k\pi, 2(k+1)\pi), k \in Z$ **2c.** $\frac{x}{2} + \frac{1}{2}\log|\sin x + \cos x|$, na $D_f (x \neq -\frac{\pi}{4} + 2k\pi, x \neq \frac{\pi}{2} + k\pi, k \in Z)$ **2d.** $x - \frac{1}{\sqrt{2}}$ arctg $(\sqrt{2}$ tg $x)$, platí na D_f mimo body $\frac{\pi}{2} + k\pi$, posun vždy o $-\frac{\pi}{\sqrt{2}}$ **2e.** $\frac{-1}{1+\mathrm{tg}\,x}$, na D_f mimo $\frac{\pi}{2} + k\pi$, lze spoj. dodef. 0 **2f.** $\frac{1}{\sqrt{5}}$ arctg $(\frac{1}{\sqrt{5}}(3\ \mathrm{tg}\ \frac{x}{2} + 1))$, mimo $\pi + 2k\pi, k \in Z$, posun vždy o $\frac{\pi}{\sqrt{5}}$ **2g.** $\frac{2}{3}$ arctg $(\frac{1}{3}(5\ \mathrm{tg}\ \frac{x}{2} + 4))$, mimo $(2k+1)\pi, k \in Z$, posun vždy o $\frac{2\pi}{3}$ **2h.** $-\frac{1}{2}($ cotg $x + \frac{1}{\sqrt{2}}$ arctg $(\frac{1}{\sqrt{2}}$ tg $x))$, mimo $\frac{\pi}{2} + k\pi, k \in Z$ **2i.** $\frac{1}{6}\log((1-\cos x)(2+\cos x)^2/(1+\cos x)^3)$, mimo $k\pi, k \in Z$ $(\equiv \frac{1}{3}\log(|t|(t^2+3))$, kde $t =$ tg $\frac{x}{2})$ **2j.** $\frac{1}{\cos x} + \frac{1}{3}\frac{1}{\cos^3 x} + \frac{1}{2}\log\sqrt{\frac{\cos x - 1}{\cos x + 1}}$ $(\equiv \frac{1}{\cos x} + \frac{1}{3}\frac{1}{\cos^3 x} + \log|\mathrm{tg}\ \frac{x}{2}|)$, mimo $\frac{k\pi}{2}, k \in Z$ **2k.** $\log(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1})$, na R

3a. $\log|\frac{1+t}{1-t}| - 2$ arctg t, kde $t = \sqrt{\frac{x-1}{1+1}}$ **3b.** $\log|x + \sqrt{x^2+1}| =$ argsinh x na R **3c.** $\log|x + \sqrt{x^2-1}|$ na int. $(-\infty,-1)$ na $(1,\infty)$ **3d.** $\log|t + \sqrt{t^2+1}|$, kde $x + \frac{1}{2} = \sqrt{\frac{3}{4}}\cdot t$ na R **3e.** $\log|t| + \frac{3}{1+2t} - \frac{3}{2}\log|1+2t|$ na R **3f.** vede na $\int \frac{1}{\sqrt{x^2+x+1}}\,dx$ **3g.** $\log|\frac{t-1}{t+1}|$, kde $\sqrt{x^2+5x+1} = x + t$, tedy $\log\frac{1}{3}\cdot\frac{\sqrt{7}}{\sqrt{7}-2}$ **3h.** $\arcsin(\frac{2}{3}(x-\frac{1}{2}))$, nebo 2arctg$\sqrt{\frac{x+1}{2-x}}$

4a. $\log\frac{t^2+2t+3}{t^2+1} + \frac{1}{\sqrt{2}}$ artcg $\frac{t+1}{\sqrt{2}} - \frac{1+t}{t^2+1} + k\frac{\pi}{\sqrt{2}}$ na int. $((2k-1)\pi,(2k+1)\pi)$, kde t =tg $\frac{x}{2}$; lze "slepit" v krajních bodech (např. $\frac{\pi}{2\sqrt{2}}$ pro $x = \pi$) **4b.** $-\log t + \log(t^2+2t+7)$, kde $t = \sqrt{x^2+7} - x$ na R **4c.** $-\frac{1}{6}\log|x-1| - \frac{1}{3(x-1)} + \frac{1}{2}\log|x+1| - \frac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}$ arctg $\frac{2x+1}{\sqrt{3}}$ na $(-\infty,-1)$ a na $(-1,1)$ a na $(1,\infty)$ **4d.** tg x+cotg $x + 2\log|\mathrm{tg}\ x - 1| - 2\log|\mathrm{tg}\ x + 1|$ na int. $(\frac{k\pi}{4}, \frac{(k+1)\pi}{4})$ **4e.** $-\frac{\sqrt{2}}{3}$ arctg $(\sqrt{2}\cos x) + \frac{1}{3\sqrt{2}}$arctg $\frac{\cos x}{\sqrt{2}}$ na R **4f.** $\frac{1}{8}\frac{1}{(\sqrt{\frac{x}{x+1}}+1)^2} + \frac{1}{8}\frac{1}{(\sqrt{\frac{x}{x+1}}-1)^2} + \frac{3}{8}\frac{1}{\sqrt{\frac{x}{x+1}}-1} - \frac{3}{8}\frac{1}{\sqrt{\frac{x}{x+1}}+1}$ na $(0,+\infty)$ **4g.** $-\frac{1}{4}\log(1+\cos x) - \frac{3}{4}\log(1-\cos x) + \frac{1}{2(\cos x - 1)}$ na $(k\pi,(k+1)\pi)$